

# Pluri-canonical system of varieties of general type

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In this paper we study when the pluri-canonical systems of a complete variety  $V$  of general type defined over the complex number field give birational mappings. The main result which will be stated below is a generalization of Maehara [2].

Let  $V$  be a complete normal Gorenstein variety with only canonical singularities and let  $K(V)$  denote a canonical divisor on  $V$ .

**Theorem.** Assume that  $K(V)$  is nef and big. Then the  $m$ -th canonical mapping  $\Phi_m$  associated with  $|mK(V)|$  gives a birational map for all  $m \geq N$ , where  $N$  depends only on  $\dim V$ .

**Proof.** To prove this, we need the following lemmas.

**Lemma 1.** Let  $V$  be a complete normal variety of dimension  $n$  and let  $r$  be  $K(V)^{(n)}$ . If  $P_m(V) \geq r m^{n-1} + n$ , then the  $m$ -th canonical mapping  $\Phi_m$  is generically finite.

This is a special case of the following lemma available for arbitrary Weil divisors.

**Lemma 1.1.** Let  $V$  be a complete normal variety over any algebraically closed field and  $D$  a Weil divisor on  $V$ . Assume  $D$  is nef and big. Let  $r$  and  $m$  be integers  $\geq 1$ .

If  $\dim H^0(V, O(mD)) > (r-1) + m^{r-1}D^n$ , then  $\dim \varphi_m(V) \geq r$ , where  $\varphi_m$  is the rational map associated with  $O(mD)$ .

**Proof.** From a classical lemma, given a subvariety  $W$  which is not contained in any hyperplane in the projective space  $\mathbf{P}^h$ , it follows that  $\deg W + \dim W > h$ . Thus we have only to prove the following lemma (cf. Matsusaka [3]).

**Lemma 1.2.** Let  $V$  be a complete normal

variety of dimension  $n$  over any algebraically closed field and  $D$  a Weil divisor on  $V$ . Assume  $D$  is nef and big. Then  $\deg \varphi_m(V) \leq m^i D^n$ , where

(i)  $\varphi_m$  is a rational map associated with  $O(mD)$ .

(ii)  $\deg \varphi_m(V)$  is the degree with respect to a hyperplane of the projective space associated with  $H^0(V, O(mD))$ , i. e.

$$\mathbf{P}(H^0(V, O(mD))).$$

(iii)  $n = \dim V$  and  $i = \dim \varphi_m(V)$

**Proof.** Let

$$\varphi_m: V \longrightarrow W = \text{Im}(\varphi_m) \subset \mathbf{P}(H^0(V, O(mD)))$$

be a rational map associated with the linear system  $|mD|$ . We take  $i$  independent divisors  $H_1, \dots, H_i$  of the complete linear system  $|O_W(1)|$ .

Here  $O_W(1)$  is the restriction  $O(1)|_W$  of the hyperplane bundle  $O(1)$  of  $\mathbf{P}(H^0(V, O(mD)))$ . Then the intersection  $H_1 \cdot H_2 \cdots H_i$  is written as a 1-cycle  $x_1 + x_2 + \cdots + x_s$ , where  $s = \deg W$ . Take a blow-up  $\mu: \tilde{V} \longrightarrow V$  such that  $\tilde{V}$  is non singular and such that  $\varphi_m \cdot \mu$  is a morphism. Denoting  $\varphi_m \cdot \mu$  by  $h$ , we have the Stein factorization  $\lambda: Z \longrightarrow W$ : then  $h$  factors through  $\lambda$  such that  $h = \lambda \cdot \phi_m$ , where  $\phi_m$  is a morphism of  $\tilde{V}$  onto  $Z$ . Clearly,

$$h^*(H_1 \cdot H_2 \cdots H_i) = h^*(H_1) \cdot h^*(H_2) \cdots h^*(H_i).$$

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Writing  $|\mu^*(O(mD))|$  in the form

$$|h^*(H_1)| + F$$

where  $F$  is the fixed component, we have

$$\begin{aligned} & (h^*(H_1 \cdots H_i), \mu^*D^{(n-i)}) \\ & \leq (h^*(H_1) + F \cdot h^*(H_2) \cdots h^*(H_i), \mu^*D^{(n-i)}) \\ & = (\mu^*mD \cdot h^*H_2 \cdots, \mu^*D^{(n-i)}), \leq m^i(\mu^*D)^{(n)} \\ & = m^iD^{(n)}. \end{aligned}$$

Putting  $\lambda^*(x_j) = y_1 + y_2 + \cdots + y_\alpha$ , where  $\alpha = \deg \lambda$ , we obtain

$$\begin{aligned} & (h^*(x_j), \mu^*D^{(n-i)}) \\ & = \mathcal{Z}_{\nu-1}^\alpha(\phi_m^*(y_\nu), \mu^*D^{(n-i)}) \\ & = \alpha(\mu^*D|_{\tilde{V}_{y_\nu}})^{(n-i)}. \end{aligned}$$

Claim 1.3.  $\gamma = (\mu^*D|_{\tilde{V}_{y_\nu}})^{(n-i)} > 0$ .

Proof of the claim. From the easy addition Theorem (Iitaka[1]),

$$\kappa(\mu^*D) \leq \kappa(\mu^*D|_{\tilde{V}_{y_\nu}}) + \dim Z.$$

Since  $\kappa(\mu^*(D)) = \dim V$ , we have  $\kappa(\mu^*D|_{\tilde{V}_{y_\nu}}) = \dim V - \dim Z$ .

This implies  $(\mu^*D|_{\tilde{V}_{y_\nu}})^{(n-i)} > 0$ . Hence  $\gamma > 0$ , which completes the proof of the claim.

Therefore

$$\begin{aligned} m^iD^{(n)} & \geq \mathcal{Z}_{j-1}^s(h^*(x_j), \mu^*D^{(n-i)}) \geq \alpha \cdot s \\ & \geq \deg \varphi_m(V). \quad \text{Q. E. D.} \end{aligned}$$

Lemma 2 (Wilson[4]). Let  $V$  be a non-singular variety of dimension  $n$ . If there exists  $m$  such that the  $m$ -th canonical mapping  $\Phi_m$  is generically finite and  $P_m(V) \geq n+2$  then  $\Phi_{nm+1}$  is birational.

Lemma 3. Let  $V$  be a complete normal Gorenstein variety of dimension  $n$  with only canonical singularities over a field of characteristic zero.

Assume that the  $m_b$ -th canonical mapping is birational.

Then the  $m$ -th canonical mapping is birational for all  $m \geq \text{Max}\{1, nm_b(m_b-1)\}$ .

Proof. For a desingularization  $\mu: V' \rightarrow V$ ,

we have  $H^0(V', O(mK_{V'})) \cong H^0(V, O(mK_V))$ . We may assume  $V$  is non-singular. Put  $W_m = \Phi_m(V)$ . Clearly,  $\text{Rat}(W_{km_b}) = \text{Rat}(W_{m_b}) = \text{Rat}(V)$  for all integers  $k \geq 1$ .

By Lemma 2,  $\text{Rat}(W_{nm_b+1}) = \text{Rat}(V)$ .

It suffices to show that there exist integers  $\alpha, \beta \geq 0$  such that  $m = \alpha(nm_b+1) + \beta m_b$ .

In fact, we have integers  $q \geq 1$ ,

$nm_b(m_b-1) > r \geq 0$  such that

$$m = qnm_b(m_b-1) + r.$$

Also,  $r = sm_b + \alpha$  for  $s \geq 0$ ,  $m_b > \alpha \geq 0$ .

Hence  $m - \alpha(nm_b+1) = \beta m_b$ , where

$$\beta = n(q(m_b-1) - \alpha) + s. \quad \text{Note that } \beta \geq 0.$$

Proof of Theorem. Let  $d: V' \rightarrow V$  be a resolution of singularities by blow-ups. From Viehweg[5],  $R^i d_* K_{V'} = 0 (i > 0)$ . Hence  $H^i(V, d_* O(K_{V'}) \otimes O(mK_V)) = 0$  for all  $i > 0$ ,  $m > 0$ . By hypothesis,  $d_* O(K_{V'}) = O(K_V)$ .

It follows that

$$\begin{aligned} P_m(V) & = \chi(V, O(mK)) \\ & = \mathcal{Z}_{i=0}^n (-1)^i \dim H^i(V, O(mK)) \\ & \quad \text{for } m \geq 2. \end{aligned}$$

Note that the leading coefficient of the polynomial  $\chi(V, O(mK))$  is equal to  $\gamma/n!$ .

Moreover if  $P_k(V) > \gamma k^{n-1} + n - 1$  (Matsusaka inequality) for a certain  $k$  such that  $2 \leq k \leq n+2$ , then we can find such a number  $N$  that all the  $m$ -th canonical mappings are birational for all  $m \geq N$  by virtue of Lemmas 1, 2 and 3.

Case 1: Assume  $\gamma \leq n-1$ . If

$P_m(V) > (n-1)(m^{n-1}+1)$ , then Matsusaka inequality holds for all  $m \in [2, n+2]$ . Hence we assume that  $P_m(V) \leq (n-1)(m^{n-1}+1)$  for all  $2 \leq m \leq n+2$ .

Then there are at most a finite number of such polynomials in the form  $P_m(V)$ .

Thus we can find a number  $\nu$  depending only on  $\dim V$  such that Matsusaka inequality holds for all  $m > \nu$ .

Case 2:  $\gamma > n-1$ .

We can assume that

$$P_m(V) < \gamma(m^{n-1}+1) \text{ for all } 2 \leq m \leq n+2.$$

If not, Matsusaka inequality holds for  $m$  with  $2 \leq m \leq n+2$  and we get the result. Under the above assumption, thus, we shall show that there exists a number  $\nu$  depending only on  $n$  such that

$P_m(V) > \gamma m^{n-1} + n - 1$  for all  $m \geq \nu$ . We construct an interpolation function  $g$  of degree  $n$  with the same leading coefficient as  $P_m(V)$  such that the polynomial equation

$P_m(V) - g(m) = 0$  has  $n-2$  roots  $< n$ , and one root  $> n$  and that  $P_m(V) > g(m)$  for all  $m \geq n+2$ . Moreover we can write  $g(m)$  as in the form  $\gamma h(m)$ , where  $h(m)$  is a polynomial in  $m$  with coefficients in  $\mathbf{Q}(n)$ . Thus,  $g(m) - \gamma(m^{n-1}+1) = 0$

is reduced to  $h(m) = m^{n-1} + 1$ . Hence easily we can find a number  $\nu$  depending only on  $n$  such that Masusaka inequality holds for all  $m \geq \nu$ . Indeed, put

$$g(i) = \gamma(i^{n-1}+1) \text{ when } i \equiv n+2 \pmod{2} \text{ and}$$

$$i \not\equiv n+2, \quad g(i) = 0 \text{ when } i \equiv n+1 \pmod{2}$$

and

$g(n+2) = \alpha\gamma$  for  $2 \leq i \leq n+2$ . Here,  $\alpha$  is determined by the following equation

$$\sum_{2 \leq i \leq n+2} g(i) / (i-2)(i-3) \cdots (\widehat{i-i}) \cdots (i-n-2) + \alpha\gamma/n! = \gamma/n!. \text{ Note that } \alpha \text{ belongs to } \mathbf{Q}(n).$$

We claim that  $P(m) > g(m)$  for all  $m \geq n+2$ , where  $P(m) = P_m(V)$ .

Consider each interval  $(i-1, i+1)$  contained in  $[2, n+1]$  for  $i \equiv n+1 \pmod{2}$ . Put  $q(m) = g(m) - P(m)$ . Then  $q(i-1) < 0$ ,  $q(i+1) < 0$  and  $q(i) = -P(i) \geq 0$

for the above  $i \equiv n+1 \pmod{2}$  by definition. Easily we know that  $q(m) = 0$  has at least two roots with multiplicities in the open interval  $(i-1, i+1)$ . Moreover, one is always in the

half interval  $(i-1, i]$  and the other in  $[i, i+1)$ , respectively. In the interval  $[2, n+1]$ , there exist  $n-1$  intervals in the form  $(i-1, i]$  or  $(i, i+1]$ . Indeed.

$$(a) \quad n \equiv 0 \pmod{2}$$

these are

$$(2, 3], [3, 4], (4, 5], \dots, (n-2, n-1], [n-1, n), (n, n+1].$$

$$(b) \quad n \equiv 1 \pmod{2}$$

$$[2, 3), (3, 4], [4, 5), \dots, (n-2, n-1], [n-1, n), (n, n+1].$$

However,  $q$  has degree  $= n-1$  by construction. Hence

$$q(m) = a(m-m_1) \cdots (m-m_{n-1}).$$

Clearly,  $m_1, \dots, m_{n-2} < n, n < m_{n-1}$ .

From  $q(n) > 0$ , it follows that  $a < 0$ .

Thus  $q(m) < 0$  for all  $m > n+1$ , i. e.

$$P(m) > g(m).$$

Hence  $P(m) > g(m) \geq \gamma(m^{n-1}+1) > \gamma m^{n-1} + n-1$

for all integers  $m \geq \text{Max}\{\text{maximal real root of } h(m) - m^{n-1} - 1 = 0, n+2\}$ .

Thus, the proof is completed.

For example,  $N = 10^{10} n^{10(n+2)}$  satisfies the required property.

## Reference

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