The Basic Metaphor of Infinity and Calculus Education

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Abstract: Why can a human brain, a collection of a finite number of cells, understand infinity? In this article, I would like to introduce G. Lakoff's cognitive method of explaining various concepts of infinity in mathematics in terms of the 'basic metaphor of infinity', and evaluate what it means in mathematics education.

Keywords: cognitive science, mathematical idea analysis, the basic metaphor of infinity

1 G. Lakoff and mathematics

G. Lakoff, a disciple of Noam Chomsky, first studied generative grammar with him, and later he moved into the study of semantics. Though young, he is by now 'classically famous' for having established various new disciplines on the fields ranging from linguistics to cognitive science, and especially, he is famous as one of the founders of cognitive linguistics. Chomsky's generative grammar once enjoyed an epoch of literally explosive progress. Then it got into stagnation, and a generation of graduate students turned to mathematics, which is a discipline inherently in close relation with linguistics, and played active parts in the fields of the foundations of mathematics, logic, and computer sciences. Many became scholars of neural computing and artificial intelligence. Perhaps it is because of these circumstances that Lakoff, a linguist in himself, has intimate relationship with many colleague mathematicians.

Lakoff himself, as he wrote in [1], has a long-standing passion for the beauty of mathematics, especially its conceptual structure. There were intelligent and passionate discussions going on between linguists, psychologists, and mathematicians at cafés and restaurants near California University. Names of seven such shops are listed at the end of the acknowledgments in [1], which are perhaps an evidence of blissful time.

Lakoff hypothesized a concept he called 'metaphor' in cognitive linguistics. Originally, metaphor means a figurative expression, a form of speech, but Lakoff's new concept of metaphor is different from this. Metaphor is an inferring mechanism functioning inside the human brain. Based on the metaphoric phenomenon in linguistic expressions, Lakoff

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hypothesized that some inference patterns, different from the inference rules in mathematics and logic, are repeatedly used in manipulating meaningful information inside the human brain.

Here are some examples:

- Business is slowing down.
- Business is turning upwards.
- I am feeling down.
- She is in high spirits.

Expressions like these, which can be seen not only in poetic phrasings but often in ordinary prose and conversation, are, as Lakoff pointed out, metaphoric. How could it happen that "business", which inherently does not have a geometric form or position, can turn upwards? In addition, we pass along these sentences from lips to lips and understand the meanings, without consciously recognizing that they are, in fact, metaphoric expressions. Our brain has the function to understand or reason about things via these kinds of metaphors, and that function is automatic. Metaphor is a concept more or less similar to the isomorphism and morphism in algebra and category theory.

Mathematics, as a discipline created by human beings, has some 'meaning' with it which cannot be reduced to mere sequence of logical symbols. We can understand mathematics not by transforming mathematical proofs into logical symbols and checking that there is no mistake applying logical rules, but by understanding the 'meaning' indicated by the theorems. Thus, the mental world of mathematics can also be explained by the function of metaphors, our brain's inferring mechanism. This is the idea shared by G. Lakoff and R. E. Núñez [1]. In order to verify their idea, they themselves took charge in a college calculus course, took part in the mathematical discussions with those students in the classroom, and examined what the problem is about mathematical conception and where the source of misunderstanding lies in learning mathematics.

Lakoff, already having applied his metaphor hypothesis on politics and gender theory with great success, had the idea from the beginning of the project that he could explain the activity of 'doing mathematics', another activity performed by human beings using the human brain, via his metaphor hypothesis.

2 BMI—the Basic Metaphor of Infinity

Let us move on to more mathematical discussion.

Take the simplest example: the concept of natural numbers. When we think of natural numbers, they are numbers 1, 2, 3, ..., and this concept of natural numbers has only meaning when there are '...' added on the tail. In another word, it deeply depends on the concept of infinity.

In the context of Christianity, humans are finite in every sense, and infinity can only be understood by God. For us, human beings, the length of life is finite, the size of our body is finite, and the number of neurons in our brain is also finite. How can a being with finite body understand infinity? This kind of discussion has been dominant in the Christian society.

On the other hand, facing the fact that such concepts so fundamental for mathematical thinking, like that of natural numbers, assume the understanding of infinity yields a firm belief that infinity is at the core of mathematical understanding. In order to understand mathematics, one has to have this concept definitely established in mind as a familiar entity. One can easily understand that merely memorizing numbers of formulas or skillfully manipulating them is far from understanding mathematics, feeling the mathematical entities as real objects, being fond of mathematics, or being interested in mathematics.

For those who understand mathematics to a certain level, 'infinity' seems to be something really innate. Not knowing where it comes from, we feel that infinity sits in our mind with real existence.

According to Lakoff and Núñez [1], one of the most important and the most impressive metaphors in mathematics is the BMI, or the Basic Metaphor of Infinity.

Lakoff assumed that there have to be some conceptual mechanism for the human brain, which is finite, to be able to feel infinity as a real existing thing, with no less existence than eggplants and cucumbers in front of our eyes. And that mechanism is the BMI.

3 Potential infinity and actual infinity

Concerning the concepts of infinity, Aristotle distinguished *potential infinity* from *actual infinity*. Potential infinity refers to the circumstance that something has no end or some action repeats itself indefinitely. Actual infinity is that kind of infinity that we feel is there as a real thing.

Potential infinity is 'a situation' which continues endlessly, whereas actual infinity is 'infinity as a thing'.

Potential infinity can be experienced in everyday life as ongoing process or motions without end. Actual infinity can never be experienced in real life; it is only a concept.

BMI is the metaphor which changes potential infinity into actual infinity. Given a situation where some operation continues endlessly, BMI will form the conceptual situation where the operation 'has been repeated an infinite number of times'.

Take the sum of sequence for example. The sum of an infinite number of terms of a sequence is symbolized by

$$\sum_{n=1}^{\infty} a_n.$$

If we change the symbol ∞ into the number 4, the formula above means the sum up to the fourth term, $a_1 + a_2 + a_3 + a_4$, but this is in fact strange, because finite sums and infinite sums are inherently distinct concepts. How can we use the same symbolic structure for different concepts? A Finite sum is a sum of several terms in the usual sense. On the other hand, an infinite sum, that is the limit of partial sums, includes an additional concept of limit.

If we take the symbol for the infinite sum 'literally' as it stands, that symbol means the sum from the first term up to the ∞^{th} term, which is absurd. Consulting a mathematics textbook, we can find the 'right' definition

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k. \qquad (1)$$

It seems that nothing uncomfortable happens as far as we obey the definition (1) strictly. But the notation of the infinite sum appearing on the left-hand side of (1) could also be interpreted as

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots + a_\infty, \qquad (2)$$

and as far as seeing this symbolization literal, (2) would be the 'right' interpretation.

Which interpretation does a mathematician take indeed? A 'classroom teacher' would interpret (1) as the limit of partial sums, very rightly, and teach his students to always keep this rule in mind and not to get confused. But for a 'working mathematician' who uses mathematical formulas for more or less practical purposes, the interpretation is different. As a practitioner, a field mathematician would interpret this formula more freely according to the situation, that is, infinite sum is sometimes a 'result of adding an infinite number of terms', a sum 'up to the ∞^{th} term', and sometimes, when he must be more careful, it is the limit. For practical purposes this ad-hoc behavior is good enough.

Literally, the ∞^{th} term does not exist. But as a reality of life, it should exist, to avoid meaningless strict calculations. It may be wrong, strictly speaking, but the symbol itself suggests the wrong usage of the symbol. And it is the workings of the BMI that allows us to imagine that something nonexistent is really existent.

So far, the reader might think that BMI is the workings of mind that takes the nonexistent term a_∞ as existent, and wonder why it is a useful mechanism in human mind. But at least for convergent series, it helps thinking, no matter whether it is well-defined mathematically.

There are many situations where BMI is useful, especially when you want to extend the range of objects in mathematical research. One good example is the hyperfunction. Hyperfunctions are literally not functions in the rigorous sense of mathematics. More important is the fact that they can be defined and treated technically and used skillfully in solving differential equations and they provide a powerful tool for engineering. Imagination of human mind is sometimes beyond mathematical rigor. Hyperfunctions are *not* functions in the usual sense, but imagination is harmless for creative thinking. This is how we, human beings, have created and extended mathematics. BMI is a powerful principle for human mathematical imagination.

4 What is mathematics?

G. Lakoff and R. E. Núñez's book [1] raised a controversial dispute among mathematicians. Especially, many comments were reported from mathematicians that the mathematical contents include many mistakes.

However, [1] is not a book of mathematics. It is a book about mathematics. It is a book on mathematical idea analysis. Inside mathematics, there is a rule that something proved from axioms using logical manipulations is called a theorem. Mathematical idea analysis tries to explain why the theorem is true, not by the proof, but by the 'meaning' of that theorem. The reason that a theorem is true is not because that theorem can be proven based on the ZFC axioms (that is, there exists a proof), but because it represents a content meaningful for human beings.

Mathematics is a creation of the human brain, and mathematical idea analysis can explain why some facts had to be treated as a theorem by human mathematics. In mathematics, if there is a theorem hard to prove, mathematicians change the axioms or change the definitions to somehow prove it. By doing so, mathematicians have extended the world of mathematics. Then, what is the mathematical world that they want to extend, paying that much effort? The answer lies not outside the fact that human beings live with human brains.

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