

A Table of a Function which appears in Bose Statistics

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A numerical table of a function appears in the formula of the radial distribution function for the ideal Bose gas is presented. It will be useful for the detailed analysis of the effects of Bose statistics on the behaviour of the system such as liquid ^4He .

1. Introduction

In the course of the theoretical study of the interacting Bose particle system such as liquid ^4He it becomes sometimes necessary to know the properties of the corresponding non-interacting system i. e. the ideal Bose gas^{1,2)}. The behaviour of the ideal Bose system is considered to disclose before us the effects of the Bose statistics which are not blurred by the presence of the inter-particle interaction. Among the various quantities concerning the system under consideration the pair correlation function—the radial distribution function—plays the central role in the theoretical calculation. It gives us the important information concerning the microscopic structure of the system. Accordingly the pair correlation function of the ideal Bose gas is quoted frequently in the course of the discussion on the real Bose system. The first detailed calculation of the pair correlation function of the ideal Bose gas has been done by F. London many years ago³⁾. His results are almost complete and have been quoted widely. But it is not given in the form convenient for the numerical use. To complement such short-

comings partly we want to give here a numerical table of the function $Q(x)$ (See (2.8)) which appears in the formula. In the section 2. the necessary definitions are given and in the section 3. the table of $Q(x)$ is given.

2. The Pair Correlation Function of the ideal Bose gas

The two-particle reduced density matrix for the ideal Bose gas of N particles in the volume V at the temperature T is defined as

$$\begin{aligned} \langle 1'2' | \rho_2^{(0)} | 12 \rangle = & \frac{\langle n_0 \rangle (\langle n_0 \rangle - 1)}{V^2} \\ & + \sum_{\substack{k_1, k_2 \\ (k_1=k_2=0)'}} f_{k_1} f_{k_2} \langle 1'2' | \mathbf{k}_1 \mathbf{k}_2 \rangle \langle \mathbf{k}_1 \mathbf{k}_2 | 12 \rangle \\ & + \langle 1'2' | \mathbf{k}_1 \mathbf{k}_2 \rangle \langle \mathbf{k}_1 \mathbf{k}_2 | 21 \rangle \end{aligned} \quad (2.1)$$

where $(\mathbf{k}_1 = \mathbf{k}_2 = 0)'$ means to omit the term $\mathbf{k}_1 = \mathbf{k}_2 = 0$ in the double summation and $\langle 12 | \mathbf{k}_1 \mathbf{k}_2 \rangle$ means the eigen function for the pair state $\mathbf{k}_1 \mathbf{k}_2$.

Here f_{k_i} ($i=1,2$) is the average number of particles in the quantum state \mathbf{k}_i i. e. the Bose distribution function

$$f_{k_i} = \frac{1}{e^{\beta(\epsilon_{k_i} - \zeta)} - 1}, \quad \epsilon_{k_i} = \frac{h^2 k_i^2}{2m}, \quad \beta = \frac{1}{kT}$$

$$k_i = |\mathbf{k}_i|$$

where m is the mass of the particle k is the Boltzmann constant and ζ is the chemical potential.

$\langle n_0 \rangle (= f_0 = 1/(e^{-\beta\zeta} - 1))$ is the average num-

* This work was done in complete collaboration with Dr. Ikuo YOSHIHARA.

(Received Sep. 10, 1984)

ver of particles in the lowest state $\mathbf{k}=0$. ζ is determined from the relation

$$N = \langle n_0 \rangle + \sum_{\mathbf{k}}' f_{\mathbf{k}} = \frac{1}{e^{-\beta\zeta} - 1} + \frac{V}{(2\pi)^3} \int d\mathbf{k} \frac{1}{e^{\beta(\epsilon_{\mathbf{k}} - \zeta)} - 1}$$

$$= \frac{1}{e^{-\beta\zeta} - 1} + \frac{V}{\lambda_T^3} \sum_{l=1}^{\infty} \frac{e^{l\beta\zeta}}{l^{3/2}},$$

$$\lambda_T^2 = \frac{h^2}{2\pi mkT} \quad (2.2)$$

As is quite well known⁴⁾ we get

$$\zeta=0, \quad \langle n_0 \rangle = N \left(1 - \left(\frac{T}{T_0} \right)^{3/2} \right), \quad \text{for } T < T_0 \quad (2.3)$$

where T_0 is the Bose-Einstein condensation temperature defined by

$$T_0 = \frac{h^2}{2\pi mk} \left(\frac{N}{V\zeta(3/2)} \right)^{3/2}, \quad \zeta\left(\frac{3}{2}\right) = \sum_{l=1}^{\infty} \frac{1}{l^{3/2}}$$

From (2.1) we get the pair correlation function $\rho_2^{(0)}(\mathbf{r}_1, \mathbf{r}_2)$ for the particles at \mathbf{r}_1 and \mathbf{r}_2 in the system as

$$\rho_2^{(0)}(\mathbf{r}_1, \mathbf{r}_2) = \frac{\langle n_0 \rangle (\langle n_0 \rangle - 1)}{V^2}$$

$$+ \frac{1}{V^2} \sum_{\substack{\mathbf{k}_1, \mathbf{k}_2 \\ (\mathbf{k}_1 = \mathbf{k}_2 = 0)'} } f_{\mathbf{k}_1} f_{\mathbf{k}_2} (1 + e^{i\mathbf{k}_1 \cdot (\mathbf{r}_1 - \mathbf{r}_2)} e^{-i\mathbf{k}_2 \cdot (\mathbf{r}_2 - \mathbf{r}_2)}) \quad (2.4)$$

This can be cast in the following form provided the necessary care is given for the Bose-Einstein condensation phenomena.

Here ρ is the particle number density and $g(r)$ is the radial distribution function.

$$\rho_2^{(0)}(\mathbf{r}_1, \mathbf{r}_2) = \rho^2 g(r), \quad \rho = \frac{N}{V}, \quad r = |\mathbf{r}_1 - \mathbf{r}_2|$$

$$g(r) = 1 + 2 \frac{\langle n_0 \rangle}{V} \rho \left(\frac{T}{T_0} \right)^{3/2} Q(\zeta, x)$$

$$+ \left(\rho \left(\frac{T}{T_0} \right)^{3/2} Q(\zeta, x) \right)^2 \quad (2.5)$$

The function $Q(\zeta, x)$ is defined through the relation

$$\sum_{\mathbf{k}} f_{\mathbf{k}} e^{i\mathbf{k} \cdot (\mathbf{r}_1 - \mathbf{r}_2)} = \frac{V}{\lambda_T^3} \sum_{l=1}^{\infty} e^{l\beta\zeta} e^{-\pi r^2 / \lambda_T^2 l} \frac{1}{l^{3/2}}$$

$$= N \left(\frac{T}{T_0} \right)^{3/2} Q(\zeta, x) \quad (2.6)$$

$$Q(\zeta, x) = \frac{1}{\zeta(3/2)} \sum_{l=1}^{\infty} e^{l\beta\zeta} e^{-x/l} \frac{1}{l^{3/2}}$$

$$x = \frac{\pi}{\lambda_T^2} r^2 \quad (2.7)$$

For $T < T_0$ we have

$$Q(x) \equiv Q(0, x) = \frac{1}{\zeta(3/2)} \sum_{l=1}^{\infty} \frac{e^{-x/l}}{l^{3/2}} \quad (2.8)$$

For the numerical calculation we use the following expression

$$Q(x) = \frac{1}{\zeta(3/2)} f(x), \quad f(x) = \sum_{l=1}^{\infty} e^{-x/l} / l^{3/2}$$

$$= e^{-x} + \sum_{n=1}^{\infty} C_n (-x)^{n-1} \quad (2.9)$$

$$C_n = \frac{\zeta(n+1/2) - 1}{(n-1)!} \quad (2.10)$$

where $\zeta(n+1/2)$ is Riemann's Zeta-function defined by

$$\zeta\left(n + \frac{1}{2}\right) = \sum_{l=1}^{\infty} \frac{1}{l^{n+1/2}}$$

In the next section we give the table of function $f(x)$ calculated from the approximate formula

$$f(x) \doteq e^{-x} + \sum_{n=1}^{20} C_n (-x)^{n-1} \quad (2.11)$$

the errors in this approximation are estimated to be less than 10^{-6} in the tabulated range. In the appendix we give a table of the ζ -function $\zeta(n+1/2)$ which is used in the calculation.

3. Table of the function $f(x)$

A Table of a Function which appears in Bose Statistics

| x | $f(x)$ | x | $f(x)$ | x | $f(x)$ | x | $f(x)$ | x | $f(x)$ |
|------|------------|------|------------|------|------------|------|------------|------|------------|
| 0.00 | 2.61237526 | 0.60 | 1.97726604 | 1.20 | 1.58122852 | 1.80 | 1.32450838 | 2.40 | 1.15066783 |
| 0.01 | 2.59901666 | 0.61 | 1.96906560 | 1.21 | 1.57601071 | 1.81 | 1.32104712 | 2.41 | 1.14826576 |
| 0.02 | 2.58576947 | 0.62 | 1.96092928 | 1.22 | 1.57083036 | 1.82 | 1.31760818 | 2.42 | 1.14587719 |
| 0.03 | 2.57263317 | 0.63 | 1.95285652 | 1.23 | 1.56568716 | 1.83 | 1.31419141 | 2.43 | 1.14350205 |
| 0.04 | 2.55960607 | 0.64 | 1.94484680 | 1.24 | 1.56058058 | 1.84 | 1.31079640 | 2.44 | 1.14114049 |
| 0.05 | 2.54668769 | 0.65 | 1.93689935 | 1.25 | 1.55551033 | 1.85 | 1.30742323 | 2.45 | 1.13879217 |
| 0.06 | 2.53387684 | 0.66 | 1.92901364 | 1.26 | 1.55047636 | 1.86 | 1.30407174 | 2.46 | 1.13645724 |
| 0.07 | 2.52117255 | 0.67 | 1.92118939 | 1.27 | 1.54547839 | 1.87 | 1.30074131 | 2.47 | 1.13413515 |
| 0.08 | 2.50857367 | 0.68 | 1.91342561 | 1.28 | 1.54051565 | 1.88 | 1.29743249 | 2.48 | 1.13182629 |
| 0.09 | 2.49607926 | 0.69 | 1.90572203 | 1.29 | 1.53558811 | 1.89 | 1.29414441 | 2.49 | 1.12953034 |
| 0.10 | 2.48368864 | 0.70 | 1.89807814 | 1.30 | 1.53069548 | 1.90 | 1.29087717 | 2.50 | 1.12724720 |
| 0.11 | 2.47140067 | 0.71 | 1.89049321 | 1.31 | 1.52583750 | 1.91 | 1.28763083 | 2.51 | 1.12497656 |
| 0.12 | 2.45921420 | 0.72 | 1.88296698 | 1.32 | 1.52101366 | 1.92 | 1.28440479 | 2.52 | 1.12271882 |
| 0.13 | 2.44712858 | 0.73 | 1.87549849 | 1.33 | 1.51622392 | 1.93 | 1.28119914 | 2.53 | 1.12047342 |
| 0.14 | 2.43514269 | 0.74 | 1.86808772 | 1.34 | 1.51146754 | 1.94 | 1.27801372 | 2.54 | 1.11824051 |
| 0.15 | 2.42325566 | 0.75 | 1.86073373 | 1.35 | 1.50674497 | 1.95 | 1.27484839 | 2.55 | 1.11601978 |
| 0.16 | 2.41146662 | 0.76 | 1.85343603 | 1.36 | 1.50205525 | 1.96 | 1.27170278 | 2.56 | 1.11381139 |
| 0.17 | 2.39977472 | 0.77 | 1.84619464 | 1.37 | 1.49739834 | 1.97 | 1.26857673 | 2.57 | 1.11161526 |
| 0.18 | 2.38817887 | 0.78 | 1.83900837 | 1.38 | 1.49277399 | 1.98 | 1.26547035 | 2.58 | 1.10943084 |
| 0.19 | 2.37667823 | 0.79 | 1.83187724 | 1.39 | 1.48818172 | 1.99 | 1.26238326 | 2.59 | 1.10725852 |
| 0.20 | 2.36527197 | 0.80 | 1.82480032 | 1.40 | 1.48362151 | 2.00 | 1.25931533 | 2.60 | 1.10509824 |
| 0.21 | 2.35395926 | 0.81 | 1.81777739 | 1.41 | 1.47909311 | 2.01 | 1.25626641 | 2.61 | 1.10294944 |
| 0.22 | 2.34273930 | 0.82 | 1.81080800 | 1.42 | 1.47459580 | 2.02 | 1.25323638 | 2.62 | 1.10081252 |
| 0.23 | 2.33161103 | 0.83 | 1.80389172 | 1.43 | 1.47012982 | 2.03 | 1.25022533 | 2.63 | 1.09868693 |
| 0.24 | 2.32057366 | 0.84 | 1.79702787 | 1.44 | 1.46569468 | 2.04 | 1.24723242 | 2.64 | 1.09657308 |
| 0.25 | 2.30962640 | 0.85 | 1.79021577 | 1.45 | 1.46129038 | 2.05 | 1.24425824 | 2.65 | 1.09447042 |
| 0.26 | 2.29876822 | 0.86 | 1.78345547 | 1.46 | 1.45691598 | 2.06 | 1.24130218 | 2.66 | 1.09237912 |
| 0.27 | 2.28799860 | 0.87 | 1.77674631 | 1.47 | 1.45257196 | 2.07 | 1.23836411 | 2.67 | 1.09029910 |
| 0.28 | 2.27731675 | 0.88 | 1.77008787 | 1.48 | 1.44825761 | 2.08 | 1.23544414 | 2.68 | 1.08823030 |
| 0.29 | 2.26672170 | 0.89 | 1.76347949 | 1.49 | 1.44397293 | 2.09 | 1.23254214 | 2.69 | 1.08617241 |
| 0.30 | 2.25621243 | 0.90 | 1.75692075 | 1.50 | 1.43971747 | 2.10 | 1.22965753 | 2.70 | 1.08412560 |
| 0.31 | 2.24578869 | 0.91 | 1.75041126 | 1.51 | 1.43549099 | 2.11 | 1.22679064 | 2.71 | 1.08208957 |
| 0.32 | 2.23544903 | 0.92 | 1.74395085 | 1.52 | 1.43129328 | 2.12 | 1.22394088 | 2.72 | 1.08006424 |
| 0.33 | 2.22519343 | 0.93 | 1.73753886 | 1.53 | 1.42712436 | 2.13 | 1.22110861 | 2.73 | 1.07804979 |
| 0.34 | 2.21502044 | 0.94 | 1.73117468 | 1.54 | 1.42298353 | 2.14 | 1.21829347 | 2.74 | 1.07604591 |
| 0.35 | 2.20492961 | 0.95 | 1.72485815 | 1.55 | 1.41887081 | 2.15 | 1.21549510 | 2.75 | 1.07405278 |
| 0.36 | 2.19492021 | 0.96 | 1.71858887 | 1.56 | 1.41478600 | 2.16 | 1.21271363 | 2.76 | 1.07206986 |
| 0.37 | 2.18499154 | 0.97 | 1.71236622 | 1.57 | 1.41072863 | 2.17 | 1.20994893 | 2.77 | 1.07009756 |
| 0.38 | 2.17514268 | 0.98 | 1.70618983 | 1.58 | 1.40669875 | 2.18 | 1.20720066 | 2.78 | 1.06813535 |
| 0.39 | 2.16537293 | 0.99 | 1.70005932 | 1.59 | 1.40269590 | 2.19 | 1.20446870 | 2.79 | 1.06618363 |
| 0.40 | 2.15568162 | 1.00 | 1.69397431 | 1.60 | 1.39871988 | 2.20 | 1.20175317 | 2.80 | 1.06424186 |
| 0.41 | 2.14606783 | 1.01 | 1.68793419 | 1.61 | 1.39477072 | 2.21 | 1.19905397 | 2.81 | 1.06231023 |
| 0.42 | 2.13653138 | 1.02 | 1.68193908 | 1.62 | 1.39084776 | 2.22 | 1.19637051 | 2.82 | 1.06038867 |
| 0.43 | 2.12707112 | 1.03 | 1.67598813 | 1.63 | 1.38695125 | 2.23 | 1.19370316 | 2.83 | 1.05847701 |
| 0.44 | 2.11768640 | 1.04 | 1.67008099 | 1.64 | 1.38308055 | 2.24 | 1.19105132 | 2.84 | 1.05657518 |
| 0.45 | 2.10837634 | 1.05 | 1.66421731 | 1.65 | 1.37923543 | 2.25 | 1.18841538 | 2.85 | 1.05468313 |
| 0.46 | 2.09914077 | 1.06 | 1.65839672 | 1.66 | 1.37541620 | 2.26 | 1.18579475 | 2.86 | 1.05280080 |
| 0.47 | 2.08997882 | 1.07 | 1.65261912 | 1.67 | 1.37162219 | 2.27 | 1.18318956 | 2.87 | 1.05092812 |
| 0.48 | 2.08088962 | 1.08 | 1.64688346 | 1.68 | 1.36785319 | 2.28 | 1.18059970 | 2.88 | 1.04906506 |
| 0.49 | 2.07187255 | 1.09 | 1.64119010 | 1.69 | 1.36410903 | 2.29 | 1.17802485 | 2.89 | 1.04721142 |
| 0.50 | 2.06292700 | 1.10 | 1.63553822 | 1.70 | 1.36038977 | 2.30 | 1.17546512 | 2.90 | 1.04536715 |
| 0.51 | 2.05405235 | 1.11 | 1.62992751 | 1.71 | 1.35669473 | 2.31 | 1.17292043 | 2.91 | 1.04353232 |
| 0.52 | 2.04524801 | 1.12 | 1.62435785 | 1.72 | 1.35302421 | 2.32 | 1.17039020 | 2.92 | 1.04170687 |
| 0.53 | 2.03651313 | 1.13 | 1.61882870 | 1.73 | 1.34937756 | 2.33 | 1.16787480 | 2.93 | 1.03989051 |
| 0.54 | 2.02784737 | 1.14 | 1.61333947 | 1.74 | 1.34575484 | 2.34 | 1.16537390 | 2.94 | 1.03808330 |
| 0.55 | 2.01924989 | 1.15 | 1.60789010 | 1.75 | 1.34215563 | 2.35 | 1.16288763 | 2.95 | 1.03628519 |
| 0.56 | 2.01072012 | 1.16 | 1.60248025 | 1.76 | 1.33858024 | 2.36 | 1.16041544 | 2.96 | 1.03449602 |
| 0.57 | 2.00225749 | 1.17 | 1.59710963 | 1.77 | 1.33502777 | 2.37 | 1.15795769 | 2.97 | 1.03271596 |
| 0.58 | 1.99386119 | 1.18 | 1.59177767 | 1.78 | 1.33149854 | 2.38 | 1.15551382 | 2.98 | 1.03094461 |
| 0.59 | 1.98553089 | 1.19 | 1.58648407 | 1.79 | 1.32799213 | 2.39 | 1.15308398 | 2.99 | 1.02918215 |

| x | $f(x)$ | x | $f(x)$ | x | $f(x)$ | x | $f(x)$ | x | $f(x)$ |
|------|------------|------|------------|------|------------|------|------------|------|------------|
| 3.00 | 1.02742841 | 3.40 | 0.96377352 | 3.80 | 0.91063066 | 4.20 | 0.86550737 | 4.60 | 0.82660936 |
| 3.01 | 1.02568347 | 3.41 | 0.96232899 | 3.81 | 0.90941362 | 4.21 | 0.86446540 | 4.61 | 0.82570456 |
| 3.02 | 1.02394704 | 3.42 | 0.96089086 | 3.82 | 0.90820142 | 4.22 | 0.86342718 | 4.62 | 0.82480279 |
| 3.03 | 1.02221917 | 3.43 | 0.95945923 | 3.83 | 0.90699416 | 4.23 | 0.86239282 | 4.63 | 0.82390402 |
| 3.04 | 1.02049984 | 3.44 | 0.95803393 | 3.84 | 0.90579169 | 4.24 | 0.86136230 | 4.64 | 0.82300826 |
| 3.05 | 1.01878887 | 3.45 | 0.95661505 | 3.85 | 0.90459423 | 4.25 | 0.86033548 | 4.65 | 0.82211560 |
| 3.06 | 1.01708633 | 3.46 | 0.95520245 | 3.86 | 0.90340152 | 4.26 | 0.85931247 | 4.66 | 0.82122582 |
| 3.07 | 1.01539205 | 3.47 | 0.95379610 | 3.87 | 0.90221354 | 4.27 | 0.85829314 | 4.67 | 0.82033888 |
| 3.08 | 1.01370611 | 3.48 | 0.95239595 | 3.88 | 0.90103038 | 4.28 | 0.85727759 | 4.68 | 0.81945503 |
| 3.09 | 1.01202822 | 3.49 | 0.95100199 | 3.89 | 0.89985192 | 4.29 | 0.85626557 | 4.69 | 0.81857401 |
| 3.10 | 1.01035858 | 3.50 | 0.94961405 | 3.90 | 0.89867812 | 4.30 | 0.85525731 | 4.70 | 0.81769594 |
| 3.11 | 1.00869702 | 3.51 | 0.94823235 | 3.91 | 0.89750909 | 4.31 | 0.85425279 | 4.71 | 0.81682081 |
| 3.12 | 1.00704338 | 3.52 | 0.94685651 | 3.92 | 0.89634456 | 4.32 | 0.85325176 | 4.72 | 0.81594848 |
| 3.13 | 1.00539773 | 3.53 | 0.94548684 | 3.93 | 0.89518476 | 4.33 | 0.85225432 | 4.73 | 0.81507907 |
| 3.14 | 1.00376003 | 3.54 | 0.94412308 | 3.94 | 0.89402943 | 4.34 | 0.85126046 | 4.74 | 0.81421245 |
| 3.15 | 1.00213011 | 3.55 | 0.94276521 | 3.95 | 0.89287867 | 4.35 | 0.85027017 | 4.75 | 0.81334861 |
| 3.16 | 1.00050794 | 3.56 | 0.94141318 | 3.96 | 0.89173245 | 4.36 | 0.84928344 | 4.76 | 0.81248767 |
| 3.17 | 0.99889358 | 3.57 | 0.94006710 | 3.97 | 0.89059077 | 4.37 | 0.84830025 | 4.77 | 0.81162948 |
| 3.18 | 0.99728689 | 3.58 | 0.93872670 | 3.98 | 0.88945347 | 4.38 | 0.84732048 | 4.78 | 0.81077406 |
| 3.19 | 0.99568780 | 3.59 | 0.93739218 | 3.99 | 0.88832067 | 4.39 | 0.84634422 | 4.79 | 0.80992150 |
| 3.20 | 0.99409618 | 3.60 | 0.93606329 | 4.00 | 0.88719222 | 4.40 | 0.84537146 | 4.80 | 0.80907156 |
| 3.21 | 0.99251220 | 3.61 | 0.93474011 | 4.01 | 0.88606822 | 4.41 | 0.84440208 | 4.81 | 0.80822446 |
| 3.22 | 0.99093560 | 3.62 | 0.93342261 | 4.02 | 0.88494854 | 4.42 | 0.84343606 | 4.82 | 0.80737997 |
| 3.23 | 0.98936645 | 3.63 | 0.93211065 | 4.03 | 0.88383316 | 4.43 | 0.84247339 | 4.83 | 0.80653820 |
| 3.24 | 0.98780460 | 3.64 | 0.93080433 | 4.04 | 0.88272206 | 4.44 | 0.84151417 | 4.84 | 0.80569925 |
| 3.25 | 0.98625012 | 3.65 | 0.92950349 | 4.05 | 0.88161522 | 4.45 | 0.84055840 | 4.85 | 0.80486288 |
| 3.26 | 0.98470273 | 3.66 | 0.92820823 | 4.06 | 0.88051263 | 4.46 | 0.83960582 | 4.86 | 0.80402920 |
| 3.27 | 0.98316264 | 3.67 | 0.92691828 | 4.07 | 0.87941426 | 4.47 | 0.83865655 | 4.87 | 0.80319808 |
| 3.28 | 0.98162969 | 3.68 | 0.92563387 | 4.08 | 0.87832011 | 4.48 | 0.83771069 | 4.88 | 0.80236964 |
| 3.29 | 0.98010384 | 3.69 | 0.92435483 | 4.09 | 0.87723003 | 4.49 | 0.83676798 | 4.89 | 0.80154286 |
| 3.30 | 0.97858506 | 3.70 | 0.92308116 | 4.10 | 0.87614413 | 4.50 | 0.83582854 | 4.90 | 0.80072063 |
| 3.31 | 0.97707318 | 3.71 | 0.92181269 | 4.11 | 0.87506239 | 4.51 | 0.83489236 | 4.91 | 0.79990004 |
| 3.32 | 0.97556841 | 3.72 | 0.92054966 | 4.12 | 0.87398468 | 4.52 | 0.83395943 | 4.92 | 0.79908211 |
| 3.33 | 0.97407035 | 3.73 | 0.91929179 | 4.13 | 0.87291109 | 4.53 | 0.83302973 | 4.93 | 0.79826657 |
| 3.34 | 0.97257921 | 3.74 | 0.91803907 | 4.14 | 0.87184150 | 4.54 | 0.83210314 | 4.94 | 0.79745366 |
| 3.35 | 0.97109483 | 3.75 | 0.91679158 | 4.15 | 0.87077600 | 4.55 | 0.83117963 | 4.95 | 0.79664325 |
| 3.36 | 0.96961730 | 3.76 | 0.91554930 | 4.16 | 0.86971434 | 4.56 | 0.83025946 | 4.96 | 0.79583546 |
| 3.37 | 0.96814635 | 3.77 | 0.91431209 | 4.17 | 0.86865675 | 4.57 | 0.82934223 | 4.97 | 0.79503004 |
| 3.38 | 0.96668217 | 3.78 | 0.91307993 | 4.18 | 0.86760309 | 4.58 | 0.82842819 | 4.98 | 0.79422722 |
| 3.39 | 0.96522449 | 3.79 | 0.91185279 | 4.19 | 0.86655334 | 4.59 | 0.82751731 | 4.99 | 0.79342688 |

Appendix. Table of ζ -function

| n | $\zeta\left(n+\frac{1}{2}\right)$ | n | $\zeta\left(n+\frac{1}{2}\right)$ |
|-----|-----------------------------------|-----|-----------------------------------|
| 1 | 2.6123753487 | 11 | 1.0003486559 |
| 2 | 1.3414872573 | 12 | 1.0001737517 |
| 3 | 1.1267338673 | 13 | 1.0000866867 |
| 4 | 1.0547075108 | 14 | 1.0000432810 |
| 5 | 1.0252045800 | 15 | 1.0000216199 |
| 6 | 1.0120058999 | 16 | 1.0000108031 |
| 7 | 1.0058267275 | 17 | 1.0000053993 |
| 8 | 1.0028592509 | 18 | 1.0000026989 |
| 9 | 1.0014125906 | 19 | 1.0000013492 |
| 10 | 1.0007008426 | 20 | 1.0000006745 |

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