

# Internal rays for polynomial skew products

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We consider Axiom A polynomial skew products on  $\mathbb{C}^2$  of degree  $d \geq 2$ . We define internal rays in the stable disks of  $W^s(\Lambda_{A_p})$  and show that they land on  $J_2$  under some assumptions.

## 1 Introduction

In this note, we consider regular polynomial skew products on  $\mathbb{C}^2$  of degree  $d \geq 2$  of the form :

$$f(z, w) = (p(z), q(z, w)).$$

If we set  $q_z(w) = q(z, w)$ , the  $n$ -th iterate of  $f$  is written by

$$f^n(z, w) = (p^n(z), Q_z^n(w)) := (p^n(z), q_{p^{n-1}(z)} \circ \cdots \circ q_z(w)).$$

Hence the dynamics on the  $z$ -plane is that of  $p$ . We call the  $z$ -plane the *base space*. The vertical planes  $\{z\} \times \mathbb{C}$  are called *fibers*. Then  $f$  preserves the family of fibers and this enables us to investigate the dynamics more precisely.

Let  $K_p$  and  $J_p$  be the *filled Julia set* and *Julia set* respectively of the polynomial  $p$  and  $A_p$  be the set of attracting periodic points of  $p$ . Let  $K$  be the set of points with bounded orbits of  $f$  and put  $K_z := \{w \in \mathbb{C}; (z, w) \in K\}$ . The *fiber Julia set*  $J_z$  is the boundary of  $K_z$ . The *second Julia set*  $J_2$ , which is a right analogue of the Julia set of a one-dimensional map, is characterized by  $J_2 = \overline{\cup_{z \in J_p} \{z\} \times J_z}$ . If  $f$  is Axiom A, then the map  $z \mapsto J_z$  is continuous in  $J_p$ , hence  $J_2 = \cup_{z \in J_p} \{z\} \times J_z$ . See Jonsson [J].

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The *stable and unstable sets* of a saddle set  $\Lambda$  are respectively defined by

$$\begin{aligned} W^s(\Lambda) &= \{y \in \mathbb{C}^2; f^n(y) \rightarrow \Lambda\}, \\ W^u(\Lambda) &= \{y \in \mathbb{C}^2; \exists \text{ prehistory } \hat{y} = (y_{-n}) \rightarrow \Lambda\}. \end{aligned}$$

Let  $\Lambda_{A_p}$  and  $\Lambda_{J_p}$  be the saddle sets in  $A_p \times \mathbb{C}$  and  $J_p \times \mathbb{C}$  respectively. Since the map  $f$  preserves the vertical fibers, it is easy to see that  $W^u(\Lambda_{A_p}) \subset A_p \times \mathbb{C}$ . Hence it follows that  $W^u(\Lambda_{A_p}) \cap W^s(\Lambda_{J_p}) = \emptyset$ . In Nakane [N], we got the following.

**Theorem 1.1.** ([N], Theorem 1.1) *Suppose that  $f$  is Axiom A. Then, the map  $z \mapsto J_z$  is continuous on  $K_p$  if and only if the property  $W^u(\Lambda_{J_p}) \cap W^s(\Lambda_{A_p}) = \emptyset$  holds.*

Let  $C_p$  and  $C_z$  ( $z \in \mathbb{C}$ ) be the set of finite critical points of  $p$  and  $q_z$  respectively. Jonsson [J] defined that a polynomial skew product  $f$  is *connected* if  $J_p$  is connected and  $J_z$  is connected for all  $z \in J_p$ , in other words,  $C_p \subset K_p$  and  $C_z \subset K_z$  for all  $z \in J_p$ .

**Theorem 1.2.** ([N], Theorem 1.2) *Suppose that  $f$  is a connected Axiom A polynomial skew product on  $\mathbb{C}^2$  with  $C_p \subset A_p$ . Then  $W^u(\Lambda_{J_p}) \cap W^s(\Lambda_{A_p}) = \emptyset$ .*

The *local stable manifold*  $W_{loc}^s(x)$  of  $x = (z_0, w_0) \in \Lambda_{A_p}$  is transversal to the fiber. That is, there exist  $\epsilon > 0$  and a holomorphic function  $\varphi(z, w_0)$  in  $\mathbb{D}(z_0, \epsilon) := \{z \in \mathbb{C}; |z - z_0| < \epsilon\}$  such that

$$W_{loc}^s(x) = \{(z, \varphi(z, w_0)); z \in \mathbb{D}(z_0, \epsilon)\}.$$

This function  $\varphi$  gives a holomorphic motion of  $J_{z_0}$  over  $\mathbb{D}(z_0, \epsilon)$ , that is,

- (1)  $\varphi(z_0, \cdot) = id_{J_{z_0}}$ ,
- (2)  $\varphi(\cdot, w)$  is holomorphic in  $\mathbb{D}(z_0, \epsilon)$  for each fixed  $w \in J_{z_0}$ ,
- (3)  $\varphi_z = \varphi(z, \cdot)$  is injective for each fixed  $z$ .

By the  $\lambda$ -lemma,  $\varphi : \mathbb{D}(z_0, \epsilon) \times J_{z_0} \rightarrow \mathbb{C}$  is continuous.

In this note, we will define internal rays in the stable disks in  $W^s(\Lambda_{A_p})$  and, under the assumptions of Theorem 1.2, we show that all internal rays land at points in  $J_2$ .

## 2 Internal rays

Under the situations in Theorem 1.2, we define the internal rays and investigate their landing on  $J_2$ .

Let  $z_0 \in A_p$  be of period  $m$  and  $U_0$  be its immediate basin. Corollary 4.3 in Roeder [R] says that, if  $f$  is connected, then the set  $C_{U_0}$  of vertical critical points of  $f$  over  $U_0$  is contained in the Fatou set of  $f$ . Hence lifting by  $f$ , the local holomorphic motion of  $J_{z_0}$  defined in the introduction extends to  $U_0$ . Thus, for  $a \in J_{z_0}$ , the connected component  $W_a$  of  $W^s(z_0, a)$  containing  $(z_0, a)$  is the graph of a holomorphic function on  $U_0$ , hence is a topological disk. We call this a *stable disk*. Put  $W_a^* = W_a \setminus \{(z_0, a)\}$ .

From the assumption  $C_p \subset A_p$ ,  $z_0$  is superattracting and we have a Böttcher coordinate at  $z_0$ . That is, there exists  $k \geq 2$  such that  $p^m(z) = z_0 + a_k(z - z_0)^k + o(|z - z_0|^k)$ ,  $a_k \neq 0$ . The Böttcher coordinate  $\phi$  at  $z_0$  is a local conformal conjugacy between  $p^m$  and the map  $\zeta \mapsto \zeta^k$ . That is, this satisfies  $\phi(z_0) = 0$  and the functional equation :

$$\phi \circ p^m(z) = \phi(z)^k.$$

It is unique up to multiplication by a  $(k-1)$ -st root of unity. By this functional equation, it is analytically continued until it meets a critical point of  $p^m$ . By the assumption  $C_p \subset A_p$ , it never meets critical points of  $p^m$ , hence it can be analytically continued to the whole basin  $U_0$  and gives a conjugacy  $\phi : U_0 \rightarrow \mathbb{D} := \mathbb{D}(0, 1)$ . Let  $\psi = \phi^{-1}$  be its inverse. This satisfies  $p^m \circ \psi(\zeta) = \psi(\zeta^k)$ .

For  $a \in J_{z_0}$ , the holomorphic motion  $\varphi(z, a)$  satisfies

$$f(z, \varphi(z, a)) = (p(z), \varphi(p(z), q_{z_0}(a)))$$

since  $f(W_{loc}^s(z_0, a)) \subset W_{loc}^s(f(z_0, a))$ . Thus it follows that

$$\begin{aligned} f^m(\psi(\zeta), \varphi(\psi(\zeta), a)) &= (p^m \circ \psi(\zeta), \varphi(p^m \circ \psi(\zeta), Q_{z_0}^m(a))) \\ &= (\psi(\zeta^k), \varphi(\psi(\zeta^k), Q_{z_0}^m(a))) \end{aligned}$$

If we define  $\psi_a : \mathbb{D} \rightarrow W_a$  by  $\psi_a(\zeta) = (\psi(\zeta), \varphi(\psi(\zeta), a))$ , it satisfies

$$f^m \circ \psi_a(\zeta) = \psi_{Q_{z_0}^m(a)}(\zeta^k). \quad (1)$$

We now define an internal ray in  $W_a$  of angle  $t \in \mathbb{R}/\mathbb{Z}$  by

$$R_a(t) = \psi_a(\{re^{2\pi it} : 0 < r < 1\}).$$

By definition,  $f^m(R_a(t)) = R_{Q_{z_0}^m(a)}(kt)$ . We say that a ray  $R_a(t)$  lands at a point  $x$  if, as  $r \rightarrow 1$ ,  $\psi_a(re^{2\pi it})$  converges to  $x$ .

**Theorem 2.1.** *Under the assumptions in Theorem 1.2, every ray  $R_a(t)$  for  $a \in J_{z_0}$  and  $t \in \mathbb{R}/\mathbb{Z}$  lands at a point in  $J_2$ . The landing points depend continuously on  $a \in J_{z_0}$  and  $t \in \mathbb{R}/\mathbb{Z}$ .*

*proof.* The proof is an analogue of that of Theorem 10.2 in [BJ]. By the uniform expansion on  $J_2$ , there exist a neighborhood  $E$  of  $J_2$ , a metric in  $E$  and  $\lambda > 1$  such that  $f^{-m}(E) \subset E$  and the following holds for all  $x \in E$  and all  $v \in T_x\mathbb{C}^2$  with this metric :

$$|Df^m(x)v|^* > \lambda|v|^*. \quad (2)$$

By the proof of Theorem 1.2 in [N],  $W_a^* \subset W^u(J_2)$  for any  $a \in J_{z_0}$ . Since the union  $A_R = \cup_{a \in J_{z_0}} \psi_a(\{R^k \leq |\zeta| \leq R\})$  is compact in  $W^u(J_2)$  for any  $R < 1$ , there exists  $n$  such that  $f^{-mn}(A_R) \subset E$ . Then there exists  $R < 1$  such that  $\psi_a(\{R \leq |\zeta| < 1\}) \subset E$  for any  $a \in J_{z_0}$ . Differentiating (1) and using (2), for  $R \leq |\zeta| < 1$ ,

$$|D\psi_a(\zeta)|^* \leq \lambda^{-1}|D\psi_{Q_{z_0}^m(a)}(\zeta^k)|^* k|\zeta|^{k-1}. \quad (3)$$

Put  $m(r) = \sup_{a \in J_{z_0}} \sup_{|\zeta|=r} |D\psi_a(\zeta)|^*$ . Take  $0 < \alpha < 1$  so that it satisfies  $k^\alpha < \lambda$  and take  $C > 0$  such that

$$m(r) \leq C(1-r)^{\alpha-1}, \quad (4)$$

holds for  $R \leq r \leq R^{1/k}$ . By (3), it follows that, for  $r \geq R$ ,

$$m(r) \leq \lambda^{-1} k r^{k-1} m(r^k). \quad (5)$$

We show, by induction on  $j$ , that (4) holds in  $R^{1/k^j} \leq r < R^{1/k^{j+1}}$  for all  $j \geq 0$ . The case  $j = 0$  is trivial. Suppose that the case  $j = n-1$  is true. If

$R^{1/k^n} \leq r < R^{1/k^{n+1}}$ , then  $R^{1/k^{n-1}} \leq r^k < R^{1/k^n}$  and we have by (5),

$$\begin{aligned}
m(r) &\leq \lambda^{-1} k r^{k-1} C (1 - r^k)^{\alpha-1} \\
&= C \lambda^{-1} k r^{k-1} (1 + r + \dots + r^{k-1})^{\alpha-1} (1 - r)^{\alpha-1} \\
&\leq C \lambda^{-1} k r^{k-1} (k r^{k-1})^{\alpha-1} (1 - r)^{\alpha-1} \\
&\leq C \lambda^{-1} k^\alpha (1 - r)^{\alpha-1} \\
&\leq C (1 - r)^{\alpha-1}.
\end{aligned}$$

Here we use the assumption  $k^\alpha < \lambda$ . Thus the case  $j = n$  is also true. Hence we get that (4) holds for  $R \leq r < 1$ .

Then, for  $\zeta_j = r_j e^{2\pi i t}$ ,  $j = 1, 2$ ,  $r_1 < r_2$ , it follows that

$$\begin{aligned}
d(\psi_a(r_1 e^{2\pi i t}), \psi_a(r_2 e^{2\pi i t})) &\leq \int_{\zeta_1}^{\zeta_2} |D\psi_a(\zeta)|^* |d\zeta| \\
&\leq \int_{r_1}^{r_2} m(r) dr \\
&\leq C \alpha^{-1} ((1 - r_1)^\alpha - (1 - r_2)^\alpha) \\
&\leq C \alpha^{-1} (r_2 - r_1)^\alpha.
\end{aligned}$$

In the last inequality, we use that  $0 < \alpha < 1$ . Thus, as  $r \rightarrow 1$ ,  $\psi_a(r e^{2\pi i t})$  converges uniformly for  $a \in J_{z_0}$  and  $t \in \mathbb{R}/\mathbb{Z}$ . This implies that every ray  $R_a(t)$  lands at a point in  $J_2$  and that the landing point depends continuously on  $a \in J_{z_0}$  and  $t \in \mathbb{R}/\mathbb{Z}$ . This completes the proof.  $\square$

**Corollary 2.1.** *The map  $\psi_a$  extends to a continuous surjective map  $\bar{\mathbb{D}} \rightarrow \bar{W}_a$ . Hence, any point in  $\partial W_a$  is the landing point of a ray. Moreover, the map  $(\zeta, a) \mapsto \psi_a(\zeta)$  is continuous on  $\bar{\mathbb{D}} \times J_{z_0}$ , the holomorphic motion  $\varphi(z, a)$  extends to a continuous function on  $\bar{U}_0 \times J_{z_0}$  and  $\varphi_z : J_{z_0} \rightarrow J_z$  is surjective for  $z \in \bar{U}_0$ .*

*proof.* We put  $\psi_a(e^{2\pi i t}) := \lim_{r \rightarrow 1} \psi_a(r e^{2\pi i t})$ . Since this convergence is uniform for  $(t, a) \in (\mathbb{R}/\mathbb{Z}) \times J_{z_0}$ , the map  $(t, a) \mapsto \psi_a(e^{2\pi i t})$  is continuous on  $(\mathbb{R}/\mathbb{Z}) \times J_{z_0}$ . For any  $\epsilon > 0$ , there exists  $\delta > 0$  such that, if  $|a' - a|, |t - s| < \delta$  and  $1 - \delta < r < 1$ ,

$$d(\psi_{a'}(r e^{2\pi i t}), \psi_a(e^{2\pi i s})) \leq d(\psi_{a'}(r e^{2\pi i t}), \psi_{a'}(e^{2\pi i t})) + d(\psi_{a'}(e^{2\pi i t}), \psi_a(e^{2\pi i s})) < 2\epsilon.$$

This implies that the map  $(\zeta, a) \mapsto \psi_a(\zeta)$  is continuous also on  $\partial\mathbb{D} \times J_{z_0}$ . Since  $\psi_a(\overline{\mathbb{D}})$  is a compact set containing  $\psi_a(\mathbb{D}) = W_a$ , it easily follows that  $\psi_a(\overline{\mathbb{D}}) = \overline{W_a}$ .

As for  $\varphi$ , note that  $\partial U_0$  is a Jordan curve. Then the map  $\psi$  extends to a homeomorphism  $\overline{\mathbb{D}} \rightarrow \overline{U_0}$ , hence the map  $\varphi(z, a) = \pi_2 \circ \psi_a \circ \psi^{-1}(z)$  extends to a continuous map on  $\overline{U_0} \times J_{z_0}$ . If we take  $z' \in \partial U_0$ , there exists a sequence  $z_n \in U_0$  tending to  $z'$ . By the continuity of the maps  $z \mapsto J_z$  and  $z \mapsto \varphi_z$  on  $\overline{U_0}$ , we have

$$J_{z'} = \lim_{n \rightarrow \infty} J_{z_n} = \lim_{n \rightarrow \infty} \varphi_{z_n}(J_{z_0}) = \varphi_{z'}(J_{z_0}).$$

Thus the equality  $\varphi_z(J_{z_0}) = J_z$  holds also for  $z \in \overline{U_0}$ .  $\square$

In general, the map  $\varphi_z : J_{z_0} \rightarrow J_z$  is not injective for  $z \in \partial U_0$ . In fact, for the following Example 2.1 with  $c = -1$ , pinching occurs as  $z \rightarrow \partial U_0$ .

**Example 2.1.**  $f(z, w) = (z^2, w^2 + cz)$ .

If we set  $g_c(w) = w^2 + c$ , then  $f^n(z, w) = (z^{2^n}, z^{2^{n-1}} g_c^n(\frac{w}{\sqrt{z}}))$ . We have  $C_p = \{0\} = A_p$ . It easily follows that  $f$  is Axiom A (resp. connected) if and only if  $g_c$  is hyperbolic (resp.  $J_{g_c}$  is connected). Thus,  $f$  satisfies the assumptions of Theorem 1.2 if  $c$  lies in a hyperbolic component of the Mandelbrot set.

Let  $\phi_c : \mathbb{C} \setminus \overline{\mathbb{D}} \rightarrow \mathbb{C} \setminus K_{g_c}$  be the inverse Böttcher coordinate of  $g_c$ . Then it follows that

$$\varphi_z(w) = \phi_z(w) = \sqrt{z} \phi_c\left(\frac{w}{\sqrt{z}}\right), \quad z \in \mathbb{D},$$

which depends holomorphically on  $z$  because  $\phi_c$  is an odd function. The internal ray  $R_a(t)$  for  $a \in J_0 = \partial\mathbb{D}$  is written as

$$R_a(t) = \{(re^{2\pi it}, \sqrt{r}e^{\pi it} \phi_c(\frac{a}{\sqrt{r}e^{\pi it}})); r < 1\}.$$

It lands at the point  $(e^{2\pi it}, e^{\pi it} \phi_c(ae^{-\pi it})) \in J_2$ . The fiber Julia set  $J_z = \phi_z(\partial\mathbb{D})$  is a Jordan curve if  $z \in \mathbb{D}$ , while it is a rotation of the Julia set  $J_c$  if  $z \in \partial\mathbb{D}$ . Thus, pinching occurs as  $z$  approaches  $\partial\mathbb{D}$ .

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