Holomorphic motion of fiber Julia sets

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We consider Axiom A polynomial skew products on $\mathbb{C}^2$ of degree $d \geq 2$. The stable manifold of a hyperbolic fiber Julia set gives a holomorphic motion of the fiber Julia set. In this note, we will show that this holomorphic motion is described by the fiberwise Böttcher coordinates.

1 Introduction

In this note, we consider regular polynomial skew products on $\mathbb{C}^2$ of degree $d \geq 2$ of the form :

$$f(z, w) = (p(z), q(z, w)).$$

If we set $q_z(w) = q(z, w)$, the k-th iterate of f is written by

$$f^k(z, w) = (p^k(z), Q_z^k(w)) := (p^k(z), q_{p^{k-1}(z)} \circ \cdots \circ q_z(w)).$$

Hence the dynamics on the z-plane is that of p. We call the z-plane *base space*. The planes $\{z\} \times \mathbb{C}$ are called *fibers*. Then f preserves the family of fibers and this enables us to investigate the dynamics.

Let K_p and J_p be the *filled Julia set* and *Julia set* respectively of the polynomial p and A_p be the set of attracting periodic points of p. Let K be the set of points with bounded orbits and put $K_z := \{w \in \mathbb{C}; (z, w) \in K\}$. The *fiber Julia set* J_z is the boundary of K_z. The *second Julia set* J_2, which is a right analogue of the Julia set of a one-dimensional map, is characterized by $J_2 = \overline{\cup_{z \in J_p} \{z\} \times J_z}$. If f is Axiom A, then the map $z \mapsto J_z$ is continuous in J_p, hence $J_2 = \cup_{z \in J_p} \{z\} \times J_z$. See Jonsson [J].

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The *stable and unstable sets* of a saddle set Λ are respectively defined by

$$\begin{aligned} W^s(\Lambda) &= \{y \in \mathbb{C}^2; f^n(y) \to \Lambda\}, \\ W^u(\Lambda) &= \{y \in \mathbb{C}^2; \exists \text{ prehistory } \hat{y} = (y_{-k}) \to \Lambda\}. \end{aligned}$$

Let $\Lambda_{A_p} = \cup_{z \in A_p}\{z\} \times J_z$ be the saddle set in $A_p \times \mathbb{C}$. Since the map f preserves the vertical fibers, it is easy to see that $W^u(\Lambda_{A_p}) \subset A_p \times \mathbb{C}$. Then the *local stable manifold* $W^s_{loc}(x)$ of $x = (z_0, w_0) \in \Lambda_{A_p}$ is transversal to the fiber. That is, there exist $\epsilon > 0$ and a holomorphic function $\varphi(z, w_0)$ in $\mathbb{D}(z_0, \epsilon)$ such that

$$W^s_{loc}(x) = \{(z, \varphi(z, w_0)); z \in \mathbb{D}(z_0, \epsilon)\}.$$

This function φ gives a holomorphic motion of J_{z_0} over $\mathbb{D}(z_0, \epsilon)$, that is,

(1) $\varphi(z_0, \cdot) = id_{J_{z_0}}$,
(2) $\varphi(\cdot, w)$ is holomorphic in $\mathbb{D}(z_0, \epsilon)$ for each fixed $w \in J_{z_0}$,
(3) $\varphi_z = \varphi(z, \cdot)$ is injective for each fixed z.

By the λ-lemma, $\varphi : \mathbb{D}(z_0, \epsilon) \times J_{z_0} \to \mathbb{C}$ is continuous.

In this note, we will show that this holomorphic motion is expressed by the fiberwise Böttcher coodrinates Φ_z. They are conformal maps in a neighborhood of the point at ∞ satisfying

$$\Phi_{p(z)} \circ q_z(w) = \Phi_z(w)^d.$$

Note that, if J_z is connected, then Φ_z extends to a conformal map $\Phi_z : \mathbb{C} \setminus K_z \to \mathbb{C} \setminus \overline{\mathbb{D}}$. Let ϕ_z be the inverse of the map Φ_z.

2 Continuation of the holomorphic motion

The following is the main theorem of this note.

Theorem 2.1. *Let f be an Axiom A polynomial skew product and $z_0 \in A_p$. Suppose that J_{z_0} is connected and that the holomorphic motion $\varphi_z : J_{z_0} \to J_z$ exists for $z \in U$ for a domain U in the immediate basin U_0 of z_0. Then $\phi_z = \varphi_z \circ \phi_{z_0}$ on $\partial\mathbb{D}$ for $z \in U$.*

Define a fiberwise external ray $R_z(\theta)$ with angle θ by

$$R_z(\theta) = \phi_z(\{re^{2\pi i\theta}; r > 1\}).$$

Then Theorem 2.1 says that, if the rays $R_{z_0}(\theta_j)$, $1 \le j \le k$, land at a same point, so do the rays $R_z(\theta_j)$, $1 \le j \le k$, for any $z \in V$. Recently Comerford and Woodard obtained a same result in [CW] for analytic families of bounded polynomial sequences.

If f is vertically expanding over K_p, we can say more : these landing properties are preserved throughout $\overline{U_0}$. As will be seen in Example 2.1, a new landing relation may appear as z approches the bounday ∂U_0.

To prove Theorem 2.1, we need a notion in Pommerenke [P]. A family $\{A_z; z \in V\}$ of compact sets in $\mathbb{C}$ is *uniformly locally connected* if, for any $\epsilon > 0$, there exists $\delta > 0$ such that for any $z \in V$ and for any $a, b \in A_z$ with $|a - b| < \delta$, there exists a connected subset $B \subset A_z$ with $a, b \in B$ and $diam\, B < \epsilon$.

Proposition 2.1. *For any compact set V in U, the family $\{J_z; z \in V\}$ is uniformly locally connected.*

Put $\psi_z = \varphi_z^{-1} : J_z \to J_{z_0}$ for $z \in U$.

Lemma 2.1. *For any $\delta_1 > 0$, there exists $\delta > 0$ such that, for any $z \in V$ and $a, b \in J_z$ with $|a - b| < \delta$, we have $|\psi_z(a) - \psi_z(b)| < \delta_1$.*

proof. We prove the lemma by contradiction. Suppose that there exists $\delta_1 > 0$ such that, for any $n \ge 1$, there exist $z_n \in V$ and $a_n, b_n \in J_{z_n}$ satisfying

$$|a_n - b_n| < 1/n, \quad |\psi_{z_n}(a_n) - \psi_{z_n}(b_n)| \ge \delta_1.$$

Put $\tilde{a}_n = \psi_{z_n}(a_n), \tilde{b}_n = \psi_{z_n}(b_n) \in J_{z_0}$. We may assume that

$$z_n \to z_\infty,\ \ a_n \to a_\infty,\ \ b_n \to b_\infty,\ \ \tilde{a}_n \to \tilde{a}_\infty,\ \ \tilde{b}_n \to \tilde{b}_\infty.$$

Then $a_\infty = b_\infty \in J_{z_\infty}$, therefore

$$\varphi_{z_\infty}(\tilde{a}_\infty) = \lim_{n\to\infty} \varphi_{z_n}(\tilde{a}_n) = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} \varphi_{z_n}(\tilde{b}_n) = \varphi_{z_\infty}(\tilde{b}_\infty).$$

This contradicts the injectivity of φ_{z_∞} because $|\tilde{a}_\infty - \tilde{b}_\infty| \geq \delta_1$. This completes the proof of Lemma 2.1. $\square$

proof of Proposition 2.1. By the equicontinuity of the family $\{\varphi_z; z \in V\}$, for any $\epsilon > 0$, there exists $\epsilon_1 > 0$ such that $diam\,\varphi_z(B) < \epsilon$ if $diam\,B < \epsilon_1$. By the local connectivity of J_{z_0}, for this ϵ_1, there exists $\delta_1 > 0$ such that, for any $\tilde{a}, \tilde{b} \in J_{z_0}$ with $|\tilde{a} - \tilde{b}| < \delta_1$, there exists a connected subset $B \subset J_{z_0}$ with $diam\,B < \epsilon_1$ containing $\tilde{a}, \tilde{b}$. By Lemma 2.1, for this δ_1, there exists $\delta > 0$ such that, for any $z \in V$ and $a, b \in J_z$ with $|a-b| < \delta$, we have $|\psi_z(a) - \psi_z(b)| < \delta_1$.

For any given $\epsilon > 0$, choose ϵ_1, δ_1 and δ as above. Then, for any $z \in V$ and for any $a, b \in J_z$ with $|a - b| < \delta$, there exists a connected set $B \subset J_{z_0}$ with $diam\,B < \epsilon_1$ containing $\psi_z(a), \psi_z(b)$. The set $\varphi_z(B) \subset J_z$ is connected, contains a, b and satisfies $diam\,\varphi_z(B) < \epsilon$. Thus the family $\{J_z; z \in V\}$ is uniformly locally connected. This completes the proof of Proposition 2.1. $\square$

proof of Theorem 2.1. Note that, for any $w \in \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$, the map $z \mapsto \phi_z(w)$ is continuous in U_0. From the assumption, J_{z_0} is locally connected, hence so is J_z for $z \in U$. By Proposition 2.1, the family $\{J_z; z \in V\}$ is uniformly locally connected. By Theorem 9.11 in [P], $\phi_z \to \phi_{z_0}$ uniformly on $\overline{\mathbb{C}} \setminus \mathbb{D}$ as $z \to z_0$.

Now, take $a = \phi_{z_0}(e^{2\pi i\theta}) \in J_{z_0}$. Then, since

$$Q_z^n \circ \phi_z(e^{2\pi i\theta}) = \phi_{z_n}(e^{2\pi i d^n\theta}), \quad z_n = p^n(z),$$

for any n, it follows that

$$d(Q_z^n \circ \phi_z(e^{2\pi i\theta}), Q_{z_0}^n(a)) = d(\phi_{z_n}(e^{2\pi i d^n\theta}), \phi_{p^n(z_0)}(e^{2\pi i d^n\theta})) \to 0.$$

Thus $(z, \phi_z(e^{2\pi i\theta})) \in W_a \cap (\{z\} \times \mathbb{C})$, hence $\phi_z(e^{2\pi i\theta}) = \varphi_z(a) = \varphi_z \circ \phi_{z_0}(e^{2\pi i\theta})$. This completes the proof of Theorem 2.1. $\square$

If f is vertically expanding over K_p, we can show a stronger result.

Corollary 2.1. *If f is vertically expanding over K_p, both functions ϕ_z and φ_z extend continuously to $z \in \overline{U_0}$, hence $\phi_z = \varphi_z \circ \phi_{z_0}$ holds for $z \in \overline{U_0}$.*

Example 2.1. $f(z, w) = (z^2, w^2 + cz)$.

If we set $g_c(w) = w^2 + c$, then $f^n(z, w) = (z^{2^n}, z^{2^{n-1}} g_c^n(\frac{w}{\sqrt{z}}))$. We have $C_p = \{0\} = A_p$. It easily follows that f is Axiom A (resp. connected) if and only if g_c is hyperbolic (resp. J_{g_c} is connected). Thus, f is vertically expanding over K_p if c lies in a hyperbolic component of the Mandelbrot set.

Let $\phi_c : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} \setminus K_{g_c}$ be the inverse Böttcher coordinate of g_c. Then it follows that

$$\varphi_z(w) = \phi_z(w) = \sqrt{z}\phi_c(\frac{w}{\sqrt{z}}), \quad z \in \mathbb{D},$$

which depends holomorphically on z because ϕ_c is an odd function. The internal ray $R_a(t)$ for $a \in J_0 = \partial\mathbb{D}$ is written as

$$R_a(t) = \{(re^{2\pi it}, \sqrt{r}e^{\pi it}\phi_c(\frac{a}{\sqrt{r e^{\pi it}}})); r < 1\}.$$

It lands at the point $(e^{2\pi it}, e^{\pi it}\phi_c(ae^{-\pi it})) \in J_2$. The fiber Julia set $J_z = \phi_z(\partial\mathbb{D})$ is a Jordan curve if $z \in \mathbb{D}$, while it is a rotation of the Julia set J_c if $z \in \partial\mathbb{D}$. Thus, pinching occurs as z approaches $\partial\mathbb{D}$. See the following figures.

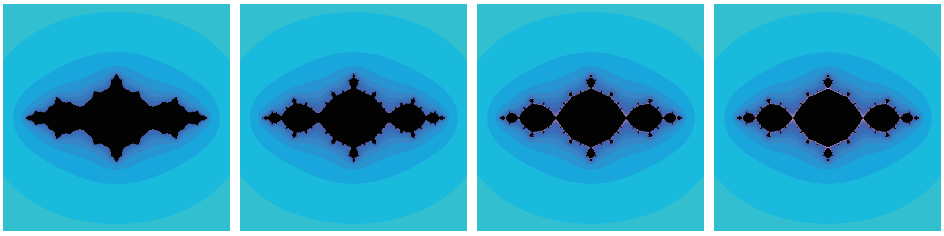

Figure 1: Fiber Julia sets ($c = -1$, from left : $z = 0.98, 0.999, 0.99999, 1$)

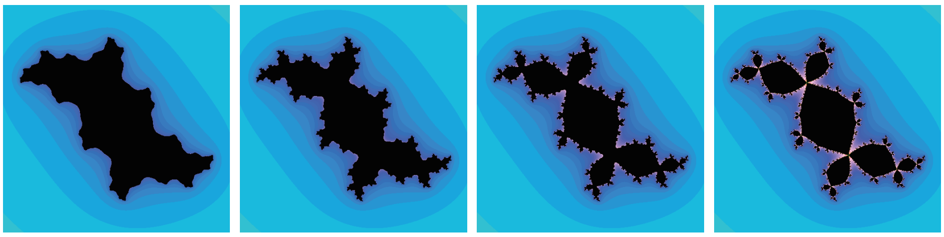

Figure 2: Fiber Julia sets (from left : $z = 0.98, 0.999, 0.99999, 1$)

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