

Remarks on Axiom A polynomial skew products on \mathbb{C}^2

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In this note, we consider Axiom A polynomial skew products on \mathbb{C}^2 of degree $d \geq 2$ with the property $W^u(\Lambda) \cap (J_p \times \mathbb{C}) = \Lambda$, which is introduced in DeMarco-Hruska [DH]. For those maps, the structure of the saddle set Λ over J_p is analyzed. It turns out that the number of saddle basic sets is equal to or less than $d - 1$. This is an analogue of the well-known fact for one-dimensional hyperbolic polynomial dynamics.

1 Introduction

In this note, we consider regular polynomial skew products on \mathbb{C}^2 of degree $d \geq 2$ of the form : $f(z, w) = (p(z), q(z, w))$. If we set $q_z(w) = q(z, w)$, the k -th iterate of f is written by

$$f^k(z, w) = (p^k(z), Q_z^k(w)) := (p^k(z), q_{p^{k-1}(z)} \circ \cdots \circ q_z(w)).$$

Hence the dynamics on the z -space is that of p . We call the z -plane *base space*. The planes $\{z\} \times \mathbb{C}$ are called *fibers*. Then f preserves the fibers and this enables us to investigate the dynamics.

Let K_p and J_p be the *filled Julia set* and *Julia set* respectively of the polynomial p and A_p be the set of attracting periodic points of p . We define the *vertical critical set* C_{J_p} of f over J_p by $C_{J_p} = \{(z, w) \in J_p \times \mathbb{C}; q'_z(w) = 0\}$. Let K be the set of points with bounded orbits and put $K_z := \{w \in \mathbb{C}; (z, w) \in K\}$. (We will use this notation L_z as a vertical slice for any subset $L \subset \mathbb{C}^2$.) The *fiber Julia set* J_z is the boundary of K_z . The *second Julia set* J_2 , which is a right analogue of the Julia set of a one-dimensional map, is characterized by $J_2 = \overline{\cup_{z \in J_p} \{z\} \times J_z}$. If f is Axiom A, then the map $z \mapsto J_z$ is continuous in J_p , hence $J_2 = \cup_{z \in J_p} \{z\} \times J_z$. See Jonsson [J]. The saddle set Λ over J_p is the hyperbolic subset of saddle type in $J_p \times \mathbb{C}$. The saddle set Λ decomposes into a disjoint union of *saddle basic sets* : $\Lambda = \sqcup_{j=1}^m \Lambda_j$, where each Λ_j is a compact invariant subset with a dense orbit.

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The *stable and unstable sets* of a saddle set Λ are respectively defined by

$$\begin{aligned} W^s(\Lambda) &= \{y \in \mathbb{C}^2; f^n(y) \rightarrow \Lambda\}, \\ W^u(\Lambda) &= \{y \in \mathbb{C}^2; \exists \text{ prehistory } \hat{y} = (y_{-k}) \rightarrow \Lambda\}. \end{aligned}$$

Those of saddle basic sets are defined similarly. In [DH], they have shown that the accumulation set $A(C_{J_p})$ of C_{J_p} is equal to $W^u(\Lambda) \cap (J_p \times \mathbb{C})$ and that $W^u(\Lambda) \cap (J_p \times \mathbb{C}) = \Lambda$ if and only if the map $z \mapsto \Lambda_z$ is continuous in J_p . In [N], the author characterized the property $W^u(\Lambda) \cap (J_p \times \mathbb{C}) = \Lambda$ in terms of the saddle basic sets. Put $C_j = C_{J_p} \cap W^s(\Lambda_j)$ for $1 \leq j \leq m$. For convenience, we add one more basic set Λ_0 , which consists of the superattracting fixed point $\{[0 : 1 : 0]\}$ in \mathbb{P}^2 and put $C_0 = C_{J_p} \cap W^s(\Lambda_0)$.

Theorem 1.1. ([N], Theorem 1.3) *For each $j \geq 1$,*

$$\begin{aligned} C_j \text{ is open in } C_{J_p} &\iff W^u(\Lambda_j) \cap (J_p \times \mathbb{C}) = \Lambda_j \\ &\iff z \mapsto \Lambda_{j,z} \text{ is continuous in } J_p. \end{aligned}$$

Consequently,

$$\begin{aligned} \forall j \geq 1, C_j \text{ is open in } C_{J_p} &\iff W^u(\Lambda) \cap (J_p \times \mathbb{C}) = \Lambda \\ &\iff z \mapsto \Lambda_z \text{ is continuous in } J_p. \end{aligned}$$

Thus, if Λ satisfies $W^u(\Lambda) \cap (J_p \times \mathbb{C}) = \Lambda$, so does Λ_j for all $j \geq 1$. Let $\pi : \mathbb{C}^2 \rightarrow \mathbb{C}$ be the projection to the base space. Then this property implies $\pi(\Lambda) = J_p$ and so on. Hence they are large.

For Axiom A maps with this property, the saddle basic sets play the role of the attracting cycles for one-dimensional hyperbolic polynomials and we will give an upper bound of the number for them. Moreover, assuming the connectivity of J_p , we will reveal the structure of saddle basic sets : the connected components of Λ are all periodic and each cycle of them forms a saddle basic set. We also give a sufficient condition for a saddle basic set to satisfy the above property.

2 Results

In this section, we will give several properties for Axiom A polynomial skew products satisfying $W^u(\Lambda) \cap (J_p \times \mathbb{C}) = \Lambda$. First, we give an upper bound of the number of saddle basic sets. This corresponds to the upper bound of the number of attracting cycles for one-dimensional hyperbolic polynomials. In fact, the proof relies on that fact.

Theorem 2.1. *Suppose f is Axiom A of degree d and satisfies $W^u(\Lambda) \cap (J_p \times \mathbb{C}) = \Lambda$. Then the number of saddle basic sets is equal to or less than $d - 1$.*

proof. By Theorem 1.1, we have $W^u(\Lambda_j) \cap (J_p \times \mathbb{C}) = \Lambda_j$ for each $j \geq 1$. This implies that $\Lambda_{j,z} \neq \emptyset$ for any $j \geq 1$ and $z \in J_p$. In particular, this is true for a fixed point $z = z_0$ of p . Any point in Λ_{j,z_0} belongs to the basin of an attracting cycle of q_{z_0} . Since $q_{z_0}(\Lambda_{j,z_0}) \subset \Lambda_{j,z_0}$, it follows that each Λ_{j,z_0} , $j \geq 1$ contains an attracting cycle. Since q_{z_0} is of degree d , it has at most $d - 1$ attracting cycles. Therefore, the number of saddle basic sets is at most $d - 1$. This completes the proof. \square

Theorem 2.1 is not true without the assumption $W^u(\Lambda) \cap (J_p \times \mathbb{C}) = \Lambda$. In fact, Theorem 7.1 in [DH] gives an example of degree two with two saddle basic sets. By the same proof as above, we have the following.

Corollary 2.1. *Suppose that f is Axiom A of degree d and $\Lambda = \sqcup_{j=1}^m \Lambda_j$ is the spectral decomposition of Λ . Then the number of Λ_j satisfying $W^u(\Lambda_j) \cap (J_p \times \mathbb{C}) = \Lambda_j$ is equal to or less than $d - 1$.*

There is a partial order \succ among $\{\Lambda_j; j \geq 0\}$ defined by

$$\Lambda_j \succ \Lambda_i \iff W^s(\Lambda_i) \cap W^u(\Lambda_j) \neq \emptyset.$$

Then the author has shown in [N] that the property $W^u(\Lambda_j) \cap (J_p \times \mathbb{C}) = \Lambda_j$ is equivalent to $W^s(\Lambda_i) \cap W^u(\Lambda_j) = \emptyset$ for any $i \neq j$, that is, Λ_j is minimal in this order.

The next theorem shows, under the assumption that J_p is connected, that the connected components of Λ have nice properties.

Theorem 2.2. *Suppose that f is Axiom A, satisfies $W^u(\Lambda) \cap (J_p \times \mathbb{C}) = \Lambda$ and J_p is connected. Then each connected component Λ' of Λ satisfies $\pi(\Lambda') = J_p$ and is periodic. That is, Λ consists of finitely many connected components, all of which are periodic.*

proof. First we show that $\pi(\Lambda') = J_p$ for any connected component Λ' of Λ . Suppose that $J' := \pi(\Lambda') \subsetneq J_p$. Since Λ' is compact, so is J' . There exists a sequence z_n in $J_p \setminus J'$ tending to $z' \in J'$ because J_p is connected. Take a point $w' \in \Lambda'_{z'}$ and put $x' = (z', w')$. Then, for any prehistory $\hat{x}' \in \hat{\Lambda}$ of x' , the local unstable manifold $W_{loc}^u(\hat{x}')$ is transversal to the vertical fiber. Hence there exist w_n for sufficiently large n such that

$$x_n = (z_n, w_n) \in W_{loc}^u(\hat{x}') \cap (J_p \times \mathbb{C}) \subset W^u(\Lambda) \cap (J_p \times \mathbb{C}) = \Lambda.$$

That is, $x_n \in \Lambda \setminus \Lambda'$ accumulates $x' \in \Lambda'$. This contradicts the fact that both sets Λ' and $\Lambda \setminus \Lambda'$ are compact.

Now we show that Λ' is periodic. Let z_0 be a fixed point of p in J_p and let A_0 be the set of attracting periodic points of q_{z_0} . Put $A_0 = \{w_j; 1 \leq j \leq K\}$ and let $\Lambda^{(j)}$ be the connected component of Λ containing (z_0, w_j) . They are periodic. Suppose

that there exists another connected component $\Lambda^{(0)}$ of Λ . If it is periodic, then $\Lambda_{z_0}^{(0)}$ contains a periodic point of q_{z_0} , which must be attracting. This contradicts the definition of $\Lambda^{(0)}$. Thus $\Lambda^{(0)}$ is not periodic. Since $f : \Lambda \rightarrow \Lambda$ is surjective, there exists a sequence $\Lambda_j^{(0)}, j \geq 0$ of connected components of Λ such that $\Lambda_0^{(0)} = \Lambda^{(0)}$ and $f(\Lambda_{j+1}^{(0)}) = \Lambda_j^{(0)}$ for $j \geq 0$. Since $\text{dist}(\Lambda, J_2) > \epsilon$ for some $\epsilon > 0$, we have $\text{dist}(\Lambda_j^{(0)}, J_2) > \epsilon$, hence $\text{dist}(\Lambda_{j,z_0}^{(0)}, J_{z_0}) > \epsilon$ for any $j \geq 0$. (Recall that, $\Lambda_{j,z_0}^{(0)} \neq \emptyset$ for any $j \geq 0$ from the first assertion.) Thus there exists a compact set K_0 in $\text{int } K_{z_0}$ such that $\cup_{j \geq 0} \Lambda_{j,z_0}^{(0)} \subset K_0$. Since $\text{int } K_{z_0}$ is the basin of A_0 , it follows that, for any $\delta > 0$, there exists N such that $q_{z_0}^n(\Lambda_{j,z_0}^{(0)})$ is included in the δ -neighborhood of A_0 for $n \geq N$ and $j \geq 0$. In particular, so is $q_{z_0}^j(\Lambda_{j,z_0}^{(0)})$ for $j \geq N$. Take $\delta > 0$ so that $\text{dist}(\Lambda^{(0)}, \Lambda^{(j)}) \geq \delta$ for any $1 \leq j \leq K$. Since $q_{z_0}^j(\Lambda_{j,z_0}^{(0)}) \subset \Lambda_{z_0}^{(0)}$, this contradicts the fact that

$$\text{dist}(q_{z_0}^j(\Lambda_{j,z_0}^{(0)}), A_0) \geq \text{dist}(\Lambda^{(0)}, \cup_{j=1}^K \Lambda^{(j)}) \geq \delta.$$

This completes the proof. \square

Generally speaking, each saddle basic set Λ_j decomposes into $\Lambda_j = \sqcup_{k=1}^{m_j} \Lambda_{j,k}$, where $\Lambda_{j,k}$ is compact and $f(\Lambda_{j,k}) = \Lambda_{j,k+1}$ for $1 \leq k \leq m_j$ with $\Lambda_{j,m_j+1} = \Lambda_{j,1}$. See Jonsson [J], Theorem A.3. The above theorem says that each $\Lambda_{j,k}$ is a connected component of Λ .

Here we give a sufficient condition so that f has a saddle basic set Λ_j satisfying $\pi(\Lambda_j) = J_p$.

Theorem 2.3. *Suppose f is Axiom A and J_z is connected for any $z \in J_p$. Then $\pi(\Lambda) = J_p$ and there exists a saddle basic set Λ_j satisfying $\pi(\Lambda_j) = J_p$.*

proof. By Proposition 2.3 in [J], J_z is connected for any $z \in J_p$ if and only if $C_{J_p} \subset K$. Take any periodic point $z_0 \in J_p$, say of period k . For any critical point (z_0, c_0) , $Q_{z_0}^{nk}(c_0)$ must tend to an attracting cycle of $Q_{z_0}^k$. Note that this attracting cycle corresponds to a saddle containing a point $(z_0, w_0) \in \Lambda$. Hence $\Lambda_z \neq \emptyset$ for any periodic point $z \in J_p$. The same holds for any $z \in J_p$ since periodic points are dense in J_p and Λ is compact. Now we conclude that $\pi(\Lambda) = J_p$.

If $\Lambda = \sqcup_{j=1}^m \Lambda_j$ is the spectral decomposition of Λ , then $\cup_{j=1}^m \pi(\Lambda_j) = \pi(\Lambda) = J_p$. Hence, for some j , $\pi(\Lambda_j)$ has non-empty interior U in J_p . Then there exists N such that $p^N(U) = J_p$. Now we have

$$\pi(\Lambda_j) = \pi(f^N(\Lambda_j)) = p^N(\pi(\Lambda_j)) = J_p.$$

This completes the proof. \square

By virtue of the property $W^u(\Lambda) \cap (J_p \times \mathbb{C}) = \Lambda$, the saddle set is reconstructed by continuing the local unstable manifolds along J_p .

Theorem 2.4. *Suppose that f is Axiom A and satisfies $W^u(\Lambda) \cap (J_p \times \mathbb{C}) = \Lambda$. Then, there exists $\epsilon > 0$ such that, for any $x = (z_0, w_0) \in \Lambda$, we have $\Lambda \cap (\mathbb{D}_\epsilon(z_0) \times \mathbb{C}) = \cup W_{loc}^u(\hat{x})$. Here the sum is taken for all prehistories of x in $\hat{\Lambda}$.*

proof. By the unstable manifold theorem, it follows that, for any prehistory $\hat{x} = (x_j)$ of x , the local unstable manifold $W_{loc}^u(\hat{x})$ exists and is transversal to the vertical fiber. Hence it is expressed by the graph of a holomorphic function $\varphi(z, \hat{x})$ on $\mathbb{D}_\epsilon(z_0)$ for some $\epsilon > 0$. Since $\hat{\Lambda}$ is compact, the radius ϵ can be taken independently of \hat{x} .

Now suppose that $y \in W_{loc}^u(\hat{x}) \cap (J_p \times \mathbb{C}) \subset \Lambda$. Then, there exists a prehistory $\hat{y} = (y_j)$ of y such that $d(y_j, x_j) < \delta$ for any $j \leq 0$. Therefore, for any $j \leq 0$, we have $y_j \in W_{loc}^u(\sigma^{-j}\hat{x}) \cap (J_p \times \mathbb{C}) \subset \Lambda$. Thus $\hat{y} \in \hat{\Lambda}$ and $x \in W_{loc}^u(\hat{y})$.

Since $\epsilon > 0$ is taken uniformly on $\hat{\Lambda}$, we can continue the local unstable manifold along J_p . By the assumption, this continuation gives Λ . Conversely, any point in Λ belongs to the local unstable manifold of a nearby point in Λ . This completes the proof. \square

References

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