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# Fibre Spaces and Groupoids

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**Abstract**

In this article we research sheaves of groupoids representation of category of algebraic normal varieties, in special a category of spectra of fields. When the ground field is algebraically closed, a spectrum of a function field with the locally algebraic automorphism group dominated by a spectrum of a trivial function field is also a spectrum of a trivial function field. Next, we propose another approach for obtaining minimal models of complex projective varieties.

## 1 Introduction

The general theory of Grothendieck Homotopy theory and étale fundamental theory give us another description of algebraic commutative and non-commutative schemes. We shall prove that if a fibre space with the generic general fibre whose automorphic groups are locally algebraic and surjectively dominated by a product fibre space, then the fibre space is also a product. This especially includes a fibre space consisted of spectra of fields. In this paper the last statement is proved on the basis that it is sufficient to recover the fields from the absolute Galois groups acting on their algebraically separable closures, i.e., groupoids instead of regarding the groups just as abstract groups. In the next section we propose a program for obtaining minimal models for complex algebraic varieties.

## 2 Preliminary

### 2.1 The theory of the homotopy type of Grothendieck[Malt],[Cisinski], [S2], [S1]

We briefly prepare an explanation of Grothendieck homotopy theory after manuscript ”Pursuing stacks” 1983 ([Malt],[Cisinski], [S2]) to apply it to small category of spectra of fields and algebraic varieties. The homotopical category $Hot$ is originally the category of the CW-complexes. The homotopical category is the category whose objects are topological spaces, and whose morphisms are homotopy equivalence classes of continuous maps. Two topological spaces X and Y are isomorphic in this category if and only if they are homotopy-equivalent. This $Hot$ is equivalent to the category localizing the category of all the topological spaces and continuous maps by topological weak equivalences $W_\infty$ such that

- $W_\infty$ is a part of the continuous
- $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ are bijections.

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- $\pi_n(f,x) : \pi_n(X,x) \to \pi_n(Y, f(x))$ are isomorphisms of groups for $n \geq 1$.

That is

$$Hot \cong W_\infty^{-1} Top.$$

The category of simplexes is the category the objects of which

$$\Delta_m = \{0, 1, \cdots, m\}$$

and the morphisms of which are increasing maps. The category $\hat{\Delta}$ of the presheaves over $\Delta$ is called the category of the simplicial sets. We have a unique couple of adjoint functors,i.e., topological realization and singular simplicial set functor

$$|\cdot| : \hat{\Delta} \to Top,\ S : Top \to \hat{\Delta}$$

A simplicial weak equivalence is a morphism of simplicial sets the topological realization of which is a homotopy equivalence. This simplicial equivalence is described $W_\infty$. Then the topological realization functor and the singular simplicial set functor are compatible with weak equivalences and induce categorical equivalences

$$W_\infty^{-1}\hat{\Delta} \to W_\infty^{-1} Top, \qquad W_\infty^{-1} Top \to W_\infty^{-1}\hat{\Delta}$$

which are quasi-inverses.

The homotopical category $Hot$ can be obtained as a localization of the 2-category $Cat$ the objects of which are the small categories and the morphisms of which are the functors. There are a couple of adjoint functors, i.e., the categorical realization functor and the nerf functor,

$$c : \hat{\Delta} \to Cat,\ N : Cat \to \hat{\Delta}$$

where the restriction of $c$ to $\Delta$ is the inclusion $\Delta \subset Cat$ and the simplicial set

$$(N(C))_m = Hom_{Cat}(\Delta_m, C),\ m \geq 0$$

the ordered set $\Delta_m$ being as a category. A categorical weak equivalence is defined to be a functor the nerf of which is a simplicial weak equivalence. Let $W_\infty$ be the part of $Fl(Cat)$ which are the categorical equivalences. The nerf functor is compatible with categorical weak equivalence and induces a categorical equivalence:

$$W_\infty^{-1} Cat \to W_\infty^{-1}\hat{\Delta}$$

However, the categorical realization functor is not compatible with the categorical weak equivalences. A quasi-inverse functor to $N$

$$Simpl : \hat{\Delta} \to Cat$$

, where it associates a simplicial set $K$ to a category of simplexes $Simpl(K)$. The $Simpl$ functor induces a categorical equivalence

$$W_\infty^{-1}\hat{\Delta} \to W_\infty^{-1} Cat$$

which is a quasi-inverse to the nerf functor $N$. A categorical weak equivalence is characterized by Artin-Mazur equivalence. Let $u : A \to B$ be a functor of small categories and $F$ a locally constant over $\hat{B}$. A morphism of toposes $(u^*, u_*) : \hat{A} \to \hat{B}$ is Artin-Mazur equivalence if and only if

$$H^m(\hat{B}, F) \to H^m(\hat{A}, u^* F)$$

are isomorphisms for $m \geq 0$ and any locally constant sheaf $F$.( as sets for $m = 0$, as groups for $m = 1$, as abelian groups for $m \geq 2$.) Let $A$ be a small category and $i_A : \hat{A} \to Cat$ $(F \to A/F)$ Then

$$W_{\hat{A}} = i_A^* W_\infty$$

A couple of adjoint functors

$$i_A : W_A^{-1}\hat{A} \to W_\infty^{-1}Cat$$

and

$$i_A^* : Hot = W_\infty^{-1}Cat \to W_{\hat{A}}^{-1}\hat{A}$$

, where

$$i_A{}^* : Cat \to \hat{A}C \mapsto (a \mapsto Hom_{Cat}(A/a, C))$$

are categorical equivalences, quasi-inverses each other if and only if $A$ is said to be a weak test category. A morphism $u : A \to B$ of $Cat$ is aspheric if $u/b : A/b \to B/b$ is a weak equivalence for any object $b$ of $B$. Grothendieck says $A$ is a weak test category if and only if for any category with a final object $i_A^*$ is aspheric, i.e., $i_A^*$ in $W_\infty$.

A category $A$ is a local test category if and only if for any object $a$ $A/a$ is a weak test category.

A category $A$ is a test category if and only if $A$ is a weak test category and a local test category at once.

A test category is as good as the category of simplicial sets $\Delta$.

**Theorem 1** *(Theorem A of Quillen's) If $u : A \to B$ is a morphism of $Cat$ such that $u/b : A/b \to B/b$ is in $W_\infty$ for any object $b$ of $B$, then $u$ is in $W_\infty$.*

**Definition 1 (Weak Basic Localizer)** *The weak basic localizer $W$ is characterized by the following conditions.*

- *(La weak saturation) the identity is in $W$. If two morphisms of a commutative triangle in $W$, then the third of them is in $W$. If $i : A \subset B$ , $r : B \to A$ and $r \circ i = id_A$, then $ir \in W$ implies $i \in W$.*
- *(Lb final object) If $A$ is a small category with a final object, then the unique functor $A \to \{pt\}$ is in $W$.*
- *(Lc Theorem A of Quillen's) If $u : A \to B$ is a morphism of $Cat$ such that $u/b : A/b \to B/b$ for all objects $b$ of $B$, then $u$ is in $W$.*

**Definition 2 (Basic Localizer)** *The basic localizer $W$ is characterized by the following conditions.*

- *(La weak saturation) the identity is in $W$. If two morphisms of a commutative triangle in $W$, then the third of them is in $W$. If $i : A \subset B$ , $r : B \to A$ and $r \circ i = id_A$, then $ir \in W$ implies $i \in W$.*
- *(Lb final object) If $A$ is a small category with a final object, then the unique functor $A \to \{pt\}$ is in $W$.*
- *(LC relative Theorem A of Quillen's) Given a commutative triangle in $Cat$*

$$\begin{tikzcd} A \arrow[rr, "u"] \arrow[dr, "v"'] & & B \arrow[dl, "w"] \\ & C & \end{tikzcd}$$

*and if $u : A \to B$ is a morphism of $Cat$ such that $u/c : A/c \to B/c$ for all objects $c$ of $C$, then $u$ is in $W$.*

Given a weak basic localizer $W$, we can do an analogue to an original categorical weak equivalence $W_\infty$. We have analogous notions, $Hot_W{=}W^{-1}Cat$, $W$-weak test categories, $W$-local test categories, $W$-test categories.

Kan extensions are universal constructs in category theory.

**Definition 3** *Let $A, B, C$ be three categories and $F : A \to B$, $G : A \to C$.*

$$\begin{tikzcd} A \arrow[r, "F"] \arrow[d, "G"'] & B \arrow[dl, bend right, "M"'] \arrow[dl, bend left, "R"] \\ C & \end{tikzcd}$$

*The right Kan extension of $G$ along $F$ consists of a functor $R : B \to C$ and a natural transformation $\eta : RF \to G$ which is co-universal so that for any functor $M : B \to C$ and a natural transformation $\mu : MF \to G$, there exists uniquely a natural transformation $\delta : M \to R$ in the following commutative diagram.*

$$\begin{tikzcd} & RF \arrow[dl, "\eta"'] & \\ G & & MF \arrow[ul, "\delta_F"'] \arrow[ll, "\mu"'] \end{tikzcd}$$

*The left Kan extension of $G$ along $F$ is dual to the right Kan extension.*

$$\begin{tikzcd} A \arrow[r, "F"] \arrow[d, "G"'] & B \arrow[dl, bend right, "L"'] \arrow[dl, bend left, "M"] \\ C & \end{tikzcd}$$

$$\begin{tikzcd} & LF \arrow[dr, "\delta_F"] & \\ G \arrow[ur, "\eta"] \arrow[rr, "\mu"'] & & MF \end{tikzcd}$$

A theory of Kan homotopical extensions is the concept of Grothendieck theory of derivaters. A localizer is a couple of $(M, W)$ such that $M$ is a category and such that $W$ is a part of morphisms $Hot_{(M,W)}(I)$ is a localization of the category $Func(I, M)$ of functors from $I$ to $M$ by induced $W$ argument-wise on natural transformations. For any functor $u : I \to J$ between small categories we have an inverse image functor $u^* : Hot_{(M,W)}(J) \to Hot_{(M,W)}(I)$

$$\begin{tikzcd} I \arrow[rr, "u"] \arrow[dr] & & J \arrow[dl] \\ & M & \end{tikzcd}$$

If there exists a structure of a Quillen,s closed model category on $M$ whose weak equivalences are $W$ and if $M$ admits small inductive limits (resp. projective limits), then there exist the left Kan extensions (resp. the right Kan extensions). For example the localizer $(Cat, W_\infty)$ is the one. For every functor $F : I \to Cat$, we obtain a cofibred category by Grothendieck construction which is called an integration of $F$ along $I$;

$$\int_I F \to I$$

From the following diagram

$$\begin{tikzcd} \int_I F \arrow[r, "u_!"] \arrow[d] & \int_J F \arrow[d] \\ I \arrow[r, "u"] & J \end{tikzcd}$$

we obtain

$$u_! : Hot_{(Cat,W)}(I) \to Hot_{(Cat,W)}(J)$$

induced by localization. This is a left adjoint functor to

$$u^* : Hot_{(Cat,W)}(J) \to Hot_{(Cat,W)}(I)$$

The dual notion of integration is co-integration, denoted by $\bigtriangledown_I F = (\int_{I^o} F^o)^o$ for a contra-variant functor $F : I^o \to Cat$.

Given a basic localizer $W$, Grothendieck says a functor between small categories $u : A \to B$ (resp. $B' \to B$ ) is $W$-proper (resp. $W$-smooth) if for any cartesian square

$$\begin{tikzcd} A' \arrow[r, "w"] \arrow[d, "u'"'] & A \arrow[d, "u"] \\ B' \arrow[r, "v"'] & B \end{tikzcd}$$

the base change morphism

$$u'_! w^* \to v^* u_!$$

is an isomorphism, where $u_! : Hot_{(Cat,W)}(A) \to Hot_{(Cat,W)}(B)$ and $u'_! : Hot_{(Cat,W)}(A') \to Hot_{(Cat,W)}(B')$ are homotopical left Kan extensions. They are dual each other.

Let $A$ be a small category and a functor

$$j_A : \hat{A} \to Cat/A$$

defined by $j_A X = (A/X, A/X \to A)$. Since $j_A^{-1}(W/A) = W_{\hat{A}}$, the functor $j_A$ induces

$$j_A : H_W \hat{A} \to Hot_W Cat//A = (W/A)^{-1} Cat/A$$

which is called the category of $W$-types of locally homotopical constants over $A$ and $H_W = W^{-1}$.

**Proposition 1** *Let $W$ a basic localizer and $A$ a $W$-local test category. Then*

$$j_A : H_W \hat{A} \to Hot_W Cat//A$$

*is a categorical equivalence.*

$$\begin{tikzcd} \hat{A} \arrow[rr, "j_A"] \arrow[d, "p^*"'] \arrow[drr, "dis"] & & Cat/A \\ A \hat{\times} \Delta & & Hom(A^o, Cat) \arrow[ll, "N"] \arrow[u, "\Xi'"'] \end{tikzcd}$$

$$\Xi' : Hom(A^o, Cat) \to Cat/A \qquad F \mapsto \bigtriangledown_A F$$

$$\Xi : Cat/A \to Hom(A^o, Cat) \qquad a \mapsto a \backslash C$$

Representations of a presheaf of small categories over a category. Let $A$ be a small category and $A$ a presheaf of small categories over a category $A$. The category $Rep(A)$ of representations of $A$ as follows. An object of $Rep(A)$ is a couple $(X, \xi)$ such that $X_a$ is a presheaf over $A_a$ for each object $a$ of $A$ and for every morphism $\alpha : a \to a'$ a morphism of presheaves $\xi_\alpha : X_{a'} \to A_a^* X_a$ for each object $a$ of $A$ and $\xi_{1_a} = 1_{X_a}$. We have a following picture;

$$\begin{array}{ccc} X_a & & X_{a'} \to A_\alpha^* X_a \\ & & \\ A_a & \xleftarrow[A_\alpha]{} & A_{a'} \\ & & \\ a & \xrightarrow{\alpha} & a' \end{array}$$

For any composed morphisms of $A$

$$a \xrightarrow{\alpha} a' \xrightarrow{\alpha'} a''$$

we have the following commutative diagram

$$\begin{tikzcd} X_{a''} \arrow[r, "\xi_\alpha"] \arrow[rr, bend right, "\xi'_\alpha \alpha"'] & A_{\alpha'}^* X_{a'} \arrow[r, "A_{\alpha'}^* \xi_\alpha"] & A_{\alpha'}^* A_\alpha^* X_a \end{tikzcd}$$

To each morphism $\phi : (X, \xi) \to (Y, \eta)$ associates a morphism $\phi_a : X_a \to Y_a$ for any object $a$ of $A$ such that for any $\alpha : a \to a'$ the following square is commutative.

$$\begin{tikzcd} X_{a'} \arrow[r, "\xi_\alpha"] \arrow[d, "\phi_{\alpha'}"'] & A_\alpha^* X_\alpha \arrow[d, "A_\alpha^* \phi_\alpha"] \\ Y_{a'} \arrow[r, "\eta_\alpha"'] & A_\alpha^* Y_\alpha \end{tikzcd}$$

Let $A$ be a small category, $G$ a presheaf of small groupoids over $A$ and $BG = \bigtriangledown G$ the classifying category associated to $G$.

**Lemma 1** *Let $A$ be a small category and* A *a presheaf of small categories over $A$. Then we obtain the canonical categorical equivalence.*

$$\hat{\bigtriangledown}\mathrm{A} \cong \mathrm{Rep(A)}$$

**Theorem 2 (Grothendieck)** *Let $W$ be a basic localizer and $u : A \to B$ a $W$-smooth functor. Then if $B$ is a $W$-local test category, $A$ is also a $W$-local test category.*

We here assume an axiom of great cardinality which is said to be the principle of Vopenka. Hence every localizer over a small category is accessible.

**Corollary 1** *Let $W$ be an accessible basic localizer and $A$ a $W$-local test category and* A *a presheaf of small categories over $A$. Then the category of* A*-representations admits a structure of closed model category with generated cofibrant the cofibrations of which are the monomorphisms and the weak equivalences of which are the $W$-equivalences. If moreover $W$ is proper, then this structure is proper. There exists a categorical equivalence between* *the homotopical category* $H_W\mathrm{Rep}(\mathrm{A})$ *and the category* $Hot_W// \bigtriangledown \mathrm{A}$ *of the locally constant* $W$*-homotopy types over* $\bigtriangledown\mathrm{A}$*.*

$$Hot_W// \bigtriangledown \mathrm{A} \cong \bigtriangledown\mathrm{A}$$

Given a presheaf of small groupoids $G$ over a small category $A$, we have a classifying space $BG$ of $G$ with a universal $G$-torsors

$$EG \to BG$$

From now on suppose a basic localizer is accessible and that $A$ is a $W$-local test category.

Since $EG$ is a $G$-torsor over $BG$ and since the morphism from $EG$ to the final object of $G$ is a right proper $W$-equivalence. We can construct such a classifying space by choosing locally cofibrant representation $EG$ that the morphism from $EG$ to the final object of $Rep(G)$ is a trivial fibration in the meaning of the structure of the category of local closed models with respect to $W$. We can hence define $BG$ as the quotient of $EG$ under action of $G$. The forgetful functor $RepG/EG \to RepG$ is a left Quillen equivalence for the categorical structures of local or injective closed models with respect to $W$.

There exists a left Quillen equivalence

$$Rep(G)/EG \to \hat{A}/BG \qquad (X, X \to EG \mapsto (G\backslash, G\backslash \to BG$$

Since we have a categorical equivalence

$$H_W Rep(G)/EG \cong H_W \hat{A}/BG$$

and a canonical categorical equivalence

$$H_W Rep(G)/EG \cong H_W Rep(G)$$

we have a descent functor which is a categorical equivalence

$$\mathbf{Desc} : H_W Rep(G) \longleftrightarrow H_W \hat{A}/BG$$

The descent functor admits a monodromy functor as quasi-inverse

$$\mathbf{Mon} : H_W \hat{A}/BG \longleftrightarrow H_W Rep(G)$$

defined by

$$(X, X \to BG \xrightarrow{Mon} X \times_{BG} EG$$

The fact that functors descent and monodromy which are quasi-inverses can be considered as a generalization of the topological Galois theory giving a correspondence between the category of covering of a locally simply connected space and the category of its fundamental groupoid representations.

## 3 Groupoids for algebraic normal varieties

Let us consider the 2-category of categories of the sheaves of groupoids representations whose objects are the sheaf of groupoids representation categories and whose morphisms are functors compatible with sheaf of groupoid action. Apply it to the category $Sch$ of locally noetherian conncted normal schemes and take a sheaf $\mathbf{G}$ of groupoids as a sheaf of categories whose objects are universal schemes over opens and whose morphisms are Grothendieck etale fundamental groups acting on universal schemes.([EGA],[SGA],[GG], [TAM],[RBZL], [Shatz]) Then

$$Rep(\mathbf{G}) = \hat{\bigtriangledown \mathbf{G}}$$

$$\mathcal{H}_W Rep(\mathbf{G}) \cong \mathcal{H}_W \hat{Sch} / \bigtriangledown \mathbf{G}$$

$$\mathcal{H}_W Rep(\mathbf{G}) \cong \mathbf{Hot}_W // \bigtriangledown \mathbf{G}$$

Let $X$ be a locally noetherian connected normal scheme and replace $Sch$ by $Sch/X$. We obtain

$$Rep(\mathbf{G}|\mathbf{X}) = \bigtriangledown \hat{\mathbf{G}}|\mathbf{X}$$

$$\mathcal{H}_W Rep(\mathbf{G}|\mathbf{X}) \cong \mathcal{H}_W \hat{Sch}/X / \bigtriangledown \mathbf{G}|\mathbf{X}$$

$$\mathcal{H}_W Rep(\mathbf{G}|\mathbf{X}) \cong \mathbf{Hot}_W // \bigtriangledown \mathbf{G}|\mathbf{X}$$

Hence $X$ is uniquely defined by $\mathbf{G}|X$. $\mathbf{G}|X$ is determined by the absolute Galois groups of all points on $X$.

**Proposition 2** *Let $k$ be a ground field. Let $K$ be a function field over $k$ and $K^s$ a separable closure of $K$ in an algebraically closed extension of $K$. Then to each automorphism $\sigma \in \mathrm{Aut}_k(K)$ associates an automorphism of $\mathrm{Gal}(K^s/K)$ such that*

$$\sigma \in \mathrm{Aut}_k(K) \longrightarrow (g \mapsto g^\sigma) \in \mathrm{Aut}(\mathrm{Gal}(K^s/K))$$

*which is a mono-morphism. Here extensions are given for each automorphism of $\mathrm{Aut}_k(K)$.*

**Proof 1** *Let $Aut_k(K^s, K) = \{g | g \in \mathrm{Aut}_k(K^s), g(K) \subset K\}$. We have a natural homomorphism*

$$\varphi : Aut_k(K^s, K) \to Aut_k(K).$$

*Then $\mathrm{Gal}(K^s/K) = \ker(\varphi)$. Note that $\mathrm{Gal}(K^s/K)$ is center-free. Thus for any $\sigma \neq id$ of $\mathrm{Aut}_k(K)$ there exists an extension $\bar{\sigma}$ since $K^s/K$ is algebraic such that $g \mapsto g^\sigma = \bar{\sigma}^{-1} g \bar{\sigma}$ is non-trivial automorphism. $g \mapsto g^\sigma = g$ for all $g$ implies $\sigma = id$. Hence $g^\alpha = g^\beta$ for all $g$ implies $\alpha = \beta$. Thus $\sigma \mapsto (g \in \mathbf{Gal}(K^s/K) \mapsto g^\sigma \in \mathbf{Gal}(K^s/K))$ is injective.([Breen1], [Breen2],[Gir], [Rou],[Se])*

**Proposition 3** *Let $G$ denote $\mathrm{Aut}_K(K^s) = \mathrm{Gal}(K^s/K)$.*

1. *there exists an exact sequence*

$$1 \to \mathrm{Aut}_K(K^s) \to \mathrm{Aut}_k(K^s, K) \to \mathrm{Aut}_k(K) \to 1$$

2. *In the following diagram, the vertical homomorphism is surjective and the horizontal homomorphism is defined by $g^\sigma$ for an element $\sigma$ of $\mathrm{Aut}_k(K^s, K)$ for any element $g$ of $\mathrm{Aut}_K(K^s)$. Here $\bar{\sigma}$ is an extension when $\sigma$ is an element of $\mathrm{Aut}_k(K)$.*

$$\sigma \mapsto \qquad (g \mapsto g^\sigma = \bar{\sigma}^{-1} g \bar{\sigma})$$

$$\begin{tikzcd} \mathrm{Aut}_k(K^s, K) \arrow[r] \arrow[d] & \mathrm{Aut}(\mathrm{Aut}_K(K^s)) \\ \mathrm{Aut}_k(K) & \end{tikzcd}$$

3.

$$\mathrm{Aut}_k(K^s) = \lim_{\longleftrightarrow_\psi} \mathrm{Aut}_k(K^s, \psi(K))$$

*where* $\psi$ *runs all* $\mathrm{Aut}_k(K^s)$.

4.

$$\mathrm{Aut}_K(K^s) \triangleleft \mathrm{Aut}_k(K^s, K)$$

5.

$$\mathrm{Aut}_{\psi(K)}(K^s) \triangleleft \mathrm{Aut}_k(K^s, \psi(K))$$

6. *There exists a surjection*

$$\mathrm{Aut}(\mathrm{Gal}(K^s/K)) \to \mathrm{Aut}_k(K^s)$$

*which is a fibre space with the same fibre* $\mathrm{Gal}(K^s/K)$.

**Proof 2** 1. *Since* $K^s/K$ *is separably algebraic, every automorphism of* $\mathrm{Aut}_k(K)$ *is able to have an extension to* $\mathrm{Aut}_k(K^s)$ *which keeps* $K$ *into itself. Hence we have a surjective homomorphism*

$$\mathrm{Aut}_k(K^s, K) \to \mathrm{Aut}_k(K)$$

*whose kernel is* $\mathrm{Aut}_K(K^s)$. *In fact* $K$ *is a fixed field by action* $\mathrm{Gal}(K^s/K)$ *on* $K^s$.

2. *From (1) it is obvious.*

3. *It follows from (1) and (2) replacing* $K$ *by* $\psi(K)$.

4. *From (1) it follows immediately.*

5. *The proof is the same argument as (4).*

6. *Since* $\mathrm{Aut}_k(\psi(K))$ *is isomorphic to* $\mathrm{Aut}_k(K)$ *and*

$$\mathrm{Aut}_k(K^s) = \lim_{\longleftrightarrow_\psi} \mathrm{Aut}_k(K^s, \psi(K))$$

*we get (6) with the aid of (1).*

**Definition 4** *Let* $k$ *be a ground field and* $K$ *a function field over* $k$. *A groupoid is defined to be a category whose object is* $\mathrm{Spec}(K^s)$ *and whose morphisms are* $\mathrm{Gal}(K^s/K)$.

**Proposition 4** *Let* $k$ *be an algebraically closed field. Let* $E$ *be a groupoid extension of a profinite group* $P$ *by a groupoid* $G$. *Assume*

- $E$ *has a group section over* $P$. *Here it is necessary to assume* $k$ *is algebraically closed.*
- $P$ *is the absolute Galois group* $\Gamma_K = \mathrm{Gal}(\hat{K}/K)$.
- $G$ *is a groupoid whose objects is the separably algebraic closure of a field in an algebraically closed field, i.e.,* $\mathrm{Spec}(K^s)$ *and whose morphisms are the absolute Galois groups acting the object.*
- *the automorphism group* $\mathrm{Aut}_{\Gamma_L}(G_L)$ *of* $G_L$ *is locally algebraic group for every algebraic extension field* $L$ *of* $K$. *([Mat], [MO]*

*Then $P$ is trivial extension after base-change by a finite extension $K'$ of $K$.*

**Proof 3**

$$\begin{tikzcd} H^1(P, Aut(G)) \arrow[r] \arrow[dr] & H^1(P, (G \to Aut(G)) \arrow[r] \arrow[d] & H^2(P, G) \\ & H^1(P, Out(G)) & \end{tikzcd}$$

*A given extension class is in $H^1(P, (G \to Aut(G))$. If $E$ has a group section, $P \to E$, $E \to Inn(E)$ and $Inn(E) \to Aut(G)$ gives an element $H^1(P, Aut(G)) \cong Hom(P, Aut(G))$. By assumption this image is a finite group. After base-change $E$ attains a trivial extension.*

We will next show a special case, which is a prototype of the precedent. Let $p$ be a prime number. Let $K$ be a subfield of a finitely generated field extension of $\mathbb{Q}_p$, which is called a sub$p$-adic field. Let $Gal(/K)$ be the absolute Galois group $Gal(\hat{K}/K)$, where $\hat{K}$ is the algebraic closure of $K$ in an algebraically closed field containing $K$. In the following commutative diagram

$$\begin{tikzcd} \mathrm{Spec}K(X) \arrow[rr] \arrow[dr] & & \mathrm{Spec}K(Y) \arrow[dl] \\ & \mathrm{Spec}K(S) \arrow[d] & \\ & \mathrm{Spec}K & \end{tikzcd}$$

where $K(X)$, $K(Y)$ and $K(S)$ are function fields which are regular extensions over $K$, i.e., $X$, $Y$, $S$ are varieties geometically irreducible, reduced over $K$, if $X$ is trivial, i.e., $X = X_0 \times_K S$ for some variety $X_0$ over $K$, when does $Y$ decompose $Y = Y_0 \times_{K'} S$ for some variety $Y_0$ over a finite extension $K'$ of $K$? The answer is yes when $\mathrm{Aut}_{\hat{K}}(\mathrm{Spec}\hat{K}(Y))$ is a group scheme which is locally of finite type, i.e., locally algebraic group.

**Remark 1**
1. *The object of our groupoid is a universal covering of a scheme to act by the absolute Galois group, where the universal covering must also a universal covering of such a scheme that we should prove in the conclusion.([Zuo]) For example it is suitable for the aim to take* Spec($K$) *as a scheme since its universal covering* Spec($K^s$) *is also a universal covering of any type sub-covering.*

2. *Let $K$ , $L$ be function fields over a ground field $k$. Assume $L$ is an extension of $K$. Let* $\mathrm{Gal}(K^s/K)$, $\mathrm{Gal}(L^s/L)$ *be the absolute Galois groups of $K$, $L$, respectively. Then we have a homomorphism* $\mathrm{Gal}(L^s/L) \to \mathrm{Gal}(K^s/K)$, *which canonically factors in the following way:*

$$\mathrm{Gal}(L^s/L) \to \mathrm{Gal}(K^s/(L \cap K^s))$$

*is a surjection.*

$$\mathrm{Gal}(K^s/(L \cap K^s)) \to \mathrm{Gal}(K^s/K)$$

*is an injection.*

3. *Non commutative analogue above is available in the case we can define the algebraically closed fields ([Cohn]).*

## 4 Program for minimal models

**Definition 5** *([MP], [Mori], [I], [Kaw], [Mats]) Let $X$ be a normal variety.*

1. *$X$ is said to be Cohen-Macaulay if each local ring $\mathcal{O}_{X,x}$ is Cohen-Macaulay for every point $x \in X$.*
2. *Let $\omega_X$ be the canonical sheaf over $X$. The $\omega_X$ is defined to be $i_*\Omega^d_{X_{reg}}$ where $i : X_{reg} \subset X$ is an inclusion and $\Omega^d_{X_{reg}}$ is a sheaf of regular $d$-forms with $d = \dim X$. It is a reflexive sheaf and a dualizing sheaf.*
3. *A canonical divisor $K_X$ is a Weil divisor associated to $\omega_X$. Further we denote*
$$\omega_X^{[m]} = i_*(\Omega^d_{X_{reg}})^{\otimes m}$$
*to which is associated a Weil divisor $mK_X$.*
4. *If $X$ is Cohen-Macaulay and there exists a number $m$ (resp. m=1) such that $mK_X$ is Cartier, $X$ is said to be $\mathbf{Q}$-Gorenstein (resp. Gorenstein).*

**Definition 6** 1. *Let $X$ be a normal $\mathbf{Q}$- variety. $X$ has only terminal (resp. canonical) singularities if $\mu_i > 0$ (resp. $\mu_i \geq 0$) for every $i$ where*
$$K_{X'} = \pi^* K_X + \sigma_i \mu_i E_i$$
*for a desingularzation $\pi : X' \to X$ and for components $E_i$ of exceptional divisors of $\pi$. $\mu_i$ is called discrepancy at $E_i$.*

2. *Let $X$ be a normal variety and $D = \sigma_i d_i D_i$ an effective $\mathbf{Q}$-Weil divisor such that $K_X + D$ is $\mathbf{Q}$-Cartier. Then $(X, D)$ is called a log pair. A desingularization $\pi : X' \to X$ is said to be a log resolution if the exceptional locus of $\pi$ and $f^{-1}(D_{red})$ have only normal crossings. We there have*
$$K_{X'} = \pi^*(K_X + D) + \sigma a_i E_i$$
*where $a_i$ are rational numbers and $E_i$ are irreducible.*
3. *$(X, D)$ is log-terminal if $d_i < 1$ and if $a_i > -1$ for all $i$.*
4. *$(X, D)$ is Kawamata log terminal, i.e., klt if $d_i \leq 1$, $a_i > -1$ and if there exists a $\pi$-ample divisor whose support is the exceptional locus.*
5. *Let $X$ be a normal ($\mathbf{Q}$-factorial) projective variety with at most terminal singularities. $X$ is said to be a minimal variety if $K_X$ is nef.*

**Conjecture 1** *Every function field has a minimal model variety if a projective model is not uni-ruled.*

Let $k$ be a complex number field. We shall propose a program in the following. Let $X$ be a projective variety. Let $k(X)$ be a function field. There exists a function field of transcendental degree $\dim X - 1$. Hence we have a fibre space between projective normal varieties
$$f : X \to S$$
such that there exist effective $\mathbf{Q}$-Cartier divisors $D_X$ and $D_S$ over $X$ and $S$, respectively such that
$$f^*(m(K_S + D_S)) \to m(K_X + D_X)$$

where $(K_S+D_S)$ and $(K_X+D_X)$ are $\mathbf{Q}$-Cartier and some $m > o$.([Mum]) To prove it we make use of Iitaka-Vieweg conjecture([Vieh], [Vieh2], [Vieh3], [Vieh4]) and its log version. In our case infinitesimal Torrelli method is useful since relative dimension is one for a fibre space $X/S$. Assume $\kappa(K_X + D_X) \geq 0$ over $X$. Then we can take $D_S$ over $S$ such that $(K_X + D_X) - f * (K_S + D_S)$ has no components pull-back of effective $\mathbf{Q}$-divisors over $S$. Then $\kappa(K_S + D_S) \geq 0$ since $\kappa(K_X + D_X) \geq 0$. By induction argument, we may take a model $S$ such that $K_S + D_S$ is nef (resp. abundant). It suffices to show

$$f_*\mathcal{O}_X(m(K_X + D_X) - f^*(m(K_S + D_S))$$

is nef (resp. abundant). We construct a fibre space $f : X \to S$ in the following way. Take a product $S \times P^1$ and an integral closure of $S \times P^1$ in the function field $k(X)$. We rename this integral closure by $X$. Thus $X$ is normal and projective. The structure map

$$\tau : X \to S \times P^1$$

is finite. There is a trace map

$$\tau_*\omega_X \to \omega_{S\times P^1}$$

If they are true, $K_X + D_X$ is nef (resp. abundant) since the tensor between $f_*\mathcal{O}_X(m(K_X + D_X) - f^*(m(K_S + D_S))$ and $f^*(m(K_S + D_S))$ is nef (resp. abundant) for some $m > 0$. Then MMC shall be a special case.

## References

[Breen1] Breen, L., *Théorie de Schreier supériure.*, Ann. scient. Éc. Norm. Sup., 4e série, t. 25, pp. 465-514(1992).

[Breen2] Breen, L., *On the classification of 2-gerbes and 2-stacks*, 225 Astérisque Société Mathématique p. 160 (1994).

[Cohn] Cohn, P.M., *Skew Fields*, Theory of general division rings, Encyclopedia of mathematics and applications 57.

[Cisinski] Cisinski, D-C., *Les préfaisceax comme modéles des types d'homotopie*, Aste risque 308, Soci et e math. France, 2006.

[Gir] Giraud, J., *Cohomologie non abélienne.*, Grundlehren math. Wiss., Springer-Verlag Berlin Heidelberg New York, p. 467,(1971).

[EGA] Grothendieck, A., Dieudonné, J., *Elements de Geometrie Algebraique I-IV*, Publications Mathematiques de IHES 4(1960),8(1961),11(1961),17(1963),20(1964),24(1965),28(1966),32(1967).

[SGA] Grothendieck, A. et al., *Sémiaire de géométrie algébrique du Bois-Marie*, SGA1, SGA4 I,II,III, SGA41/2, SGA5, SGA7 I,II, Lecture Notes in Mat., vols. 224,269-270-305,569,589,288-340, Springer-Verlag, New York, 1971-1977.

[GG] Grothendieck, A., *Fondaments de la géométrie algébrique.*, Secrétrariat mathématique, 11 rue Pierre Curis, Paris 5e, p. 236 (1962).

[I] Iitaka, S., *Introduction to birational geometry.*, Graduate Textbook in Mathematics, Springer-Verlag, p. 357 (1976).

[Kaw] Kawamata, Y., *Minimal models and the Kodaira dimension of algebraic fibre spaces.*, J. Reine Angew. Math. 363, pp. 1-46 (1985).

[Kon] Kontsevich, M., Rosenberg A. *Noncommutative smooth spaces.*, arXiv:math.AG/9812158v1 30Dec 1998.

[Malt] Maltsiniotis, G. *La th eorie de l'homotopie de Grothendieck*, Aste risque 301, Soci et e math. France, 2005.

[Mats] Matsuki, K., Introduction to the Mori Program., Universitext p. 468 Springer 2000

[Mat] Matsumura, H., On algebraic groups of birational transformations.,Rend. Accad. Naz. Lincei, Serie VIII,34, 151-155 (1963).

[MO] Matsumura, H.and Oort, F., Representability of group functors, and automorphisms of algebraic schemes, Inventiones Mathematicae, 1967 - Springer.

[MP] Miyaoka, Y., Peternel T., Geometry of Higher Dimensional Algebraic Varieties., DMV Seminar Band 26 Birkhäiser p. 213 1997

[Mch] Mochizuki, S., The local Pro-$p$ Anabelian Geometry of Curves., Research Institute for Mathematical Sciences, Kyoto University RIMS-1097(1996).

[Mori] Mori,S., Classification of higher-dimensional varieties. Algebraic Geometry Bowdoin 1985. Proc. Symp. Pure Math. 46, 269-231(1987).

[Mum] Mumford D., Selected Papers: On the classification of varieties and moduli spaces., Springer p. 795 2003

[Rou] Rousseau, A., Bicatégories monoidales et extensions de gr-categories, Homology, Homotopy and Applications, vol.5(1), 2003, pp.437-547.

[RBZL] Ribes, L., Zalesskii, P., Profinite Groups., Ergebnisse der Mathematik und ihrer Grenzgebiete 3.Folge Vol.40 Springer(1991).

[TAM] SZAMUERY, T., *Galois Groups and Fundamental Groups*, Cambridge studies in advanced mathematics 117(2009).

[Se] J-P. Serre. Cohomologie Galoisienne. Lecture Notes in Mathematics. , **5** 4th Ed. 1973.

[S1] Simpson, C., *Algebraic (geometric) n-stacks.*, arXiv:math. AG/9609014

[S2] Simpson, C., *A closed model structure for n-categories, internal Hom, n-stacks and generalized Seifer-Van Kampen.*, arXiv:math: AG/9704006 v2 17 Mar 2000

[Shatz] Shatz, S., *Profinite groups, arithmetic, and geometry.*, Annals of Math. studies, N. 67, Princeton Uni. Press, p. 252,(1972).

[Vieh] Viehweg, E. *Quasi-projective Moduli for Polarized Manifolds.*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3.Folge.Band 30, p. 320 (1991).

[Vieh2] Viehweg, E., *Weak positivity and the additivity of the Kodaira dimension for certain fibre spaces.* Algebraic Varieties and Analytic varieties. (Adv.Stud. Pure Math., Vol.39, pp.329-353). North-Holland 1983.

[Vieh3] Viehweg, E., *Weak positivity and the additivity of the Kodaira dimension II; The local Torelli map.* Classification of Algebraic and Analytic Manifolds.(Progress in Math,. Vol.39, pp.567-589). Boston: Birkhaeuser 1983.

[Vieh4] Viehweg, E., *Weak positivity and the stability of certain Hilbert points.I,II,III*, Invent.Math.96,639-667(1989),Invent.Math.101, 191-223(1990),Invent. Math. 101, 521-543(1990).

[Zuo] Zuo, K., Representations of fundamental groups of algebraic varieties., Lecture Notes in Math. 1708, Springer, ISBN 3-540-66312-6.