

# An example of non Axiom A polynomial skew products

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In this note, the dynamics of a non Axiom A polynomial skew product on  $\mathbb{C}^2$  is investigated. It follows that there exist countably many saddle periodic points in the second Julia set, which never occurs for Axiom A maps.

## 1 Introduction

In this note, we consider a regular polynomial skew product on  $\mathbb{C}^2$  of the form :

$$f(z, w) = (p(z), q(z, w)) = (z^2, w^2 + 2(1 - z)w).$$

It was first studied in Jonsson [J]. He has shown that it has a saddle fixed point  $(1, 0)$  which lies in the second Julia set  $J_2$ . Consequently, it is not Axiom A. His argument also says that  $J_p \times \{0\} \subset J_2$ . Thus all saddle periodic points of the form  $(z, 0)$  lie in  $J_2$ .

We will show that there exist countably many saddle periodic points of the form  $(z, 0)$ , which belong to  $J_2$  and that some of those saddles contain vertical critical points in their stable sets. It turns out that the corresponding fiber Julia sets are disconnected except  $z = 1$ .

## 2 Saddle periodic points in $J_2$

Here is the main theorem.

**Theorem 2.1.** *Let  $z = e^{2\pi it}$  with  $t = \frac{m}{2^k - 1}$  be a periodic point of  $p$  of period  $k$ . If  $4m^2\pi^2 + 1 < 2^{k-1}$ , then the point  $(z, 0)$  is a saddle periodic point of  $f$  of period  $k$  which lies in  $J_2$ .*

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*proof.* Set  $q_z(w) = q(z, w)$  and  $Q_z^n(w) = q_{p^{n-1}(z)} \circ \cdots \circ q_z(w)$ . Since  $q_z(0) = 0$ , the point  $(z, 0)$  is periodic for  $f$  if so is  $z$ . We have only to show that  $|(Q_z^k)'(0)| < 1$ . If we set  $z_j = p^j(z)$ , we have

$$(Q_z^k)'(0) = \prod_{j=0}^{k-1} q'_{z_j}(0) = \prod_{j=0}^{k-1} 2(1 - z_j).$$

Since

$$|1 - e^{2\pi it}| = \sqrt{(1 - \cos 2\pi t)^2 + \sin^2 2\pi t} = \sqrt{2(1 - \cos 2\pi t)} = 2|\sin \pi t|,$$

and  $|\sin x| \leq x$  for  $x \geq 0$ , we have

$$\begin{aligned} |(Q_z^k)'(0)| &= \prod_{j=0}^{k-1} 4|\sin 2^j \pi t| \\ &\leq \prod_{j=0}^{k-1} 4 \cdot \frac{2^j m \pi}{2^k - 1} \\ &= \left( \frac{4m\pi}{2^k - 1} \right)^k 2^{k(k-1)/2} \\ &= \left( \frac{2^{(k+3)/2} m \pi}{2^k - 1} \right)^k. \end{aligned}$$

Thus the point  $(z, 0)$  is a saddle if

$$\frac{2^{(k+3)/2} m \pi}{2^k - 1} < 1 \quad \text{that is, } 2^{(k+3)/2} m \pi < 2^k - 1.$$

It holds if  $2^{k+3} m^2 \pi^2 < 2^{2k} - 2^{k+1}$  i.e.  $4m^2 \pi^2 + 1 < 2^{k-1}$ . This completes the proof.  $\square$

In case  $m = 1$ ,  $2^{k-1} > 4\pi^2 + 1$  if and only if  $k \geq 7$ . On the other hand, we can show  $|(Q_z^k)'(0)| > 1$  if  $2 \leq k \leq 5$ . Thus Theorem 2.1 is good in this sense.

**Theorem 2.2.** *Let  $z = e^{2\pi it}$  with  $t = \frac{1}{2^k - 1}$  be a periodic point of  $p$  of period  $k$ . If  $2 \leq k \leq 5$ , then the point  $(z, 0)$  is a repelling periodic point of  $f$  of period  $k$ .*

*proof.* We will estimate

$$\lambda_k = (Q_z^k)'(0) = \prod_{j=0}^{k-1} 4 \sin \frac{2^j \pi}{2^k - 1}.$$

If  $k = 2$ ,  $\lambda_2 = 16 \sin \frac{\pi}{3} \sin \frac{2\pi}{3} = 12$ .

In case  $k = 3$ , we use the inequality :  $\sin x \geq \frac{x}{2}$  for  $0 \leq x \leq \frac{\pi}{3}$ . Since  $\frac{2\pi}{7} \leq \frac{\pi}{3}$ , we have

$$\lambda_3 = 4^3 \sin \frac{\pi}{7} \sin \frac{2\pi}{7} \cdot 2 \sin \frac{2\pi}{7} \cos \frac{2\pi}{7} \geq 4^3 \cdot \frac{\pi}{14} \frac{\pi}{7} \cdot 2 \frac{\pi}{7} \cdot \frac{1}{2} = \frac{4^3 \pi^3}{2 \cdot 7^3} \geq \frac{12^3}{2 \cdot 7^3} > 1.$$

In case  $k = 4$ , we use the inequality :  $\sin x \geq \frac{\sqrt{3}x}{2}$  for  $0 \leq x \leq \frac{\pi}{6}$ . Since  $\frac{2\pi}{15} \leq \frac{\pi}{6} \leq \frac{4\pi}{15} \leq \frac{\pi}{3}$ ,

$$\begin{aligned} \lambda_4 &= 4^4 \sin \frac{\pi}{15} \sin \frac{2\pi}{15} \sin \frac{4\pi}{15} \cdot 2 \sin \frac{4\pi}{15} \cos \frac{4\pi}{15} \\ &\geq 4^4 \frac{\sqrt{3}\pi}{2 \cdot 15} \frac{2\sqrt{3}\pi}{2 \cdot 15} \frac{4\pi}{2 \cdot 15} \cdot 2 \frac{4\pi}{2 \cdot 15} \frac{1}{2} = \frac{4^5 \cdot 3\pi^4}{2 \cdot 15^4} = 6 \left( \frac{4\pi}{15} \right)^4 \geq 6 \left( \frac{4}{5} \right)^4 > 1. \end{aligned}$$

In case  $k = 5$ , we use the inequality :  $\sin x \geq \frac{(\sqrt{6} + \sqrt{2})x}{4}$  for  $0 \leq x \leq \frac{\pi}{12}$ . Since  $\frac{2\pi}{31} \leq \frac{\pi}{12} \leq \frac{4\pi}{31} \leq \frac{\pi}{6} \leq \frac{8\pi}{31} \leq \frac{\pi}{3}$ ,

$$\begin{aligned} \lambda_5 &= 4^5 \sin \frac{\pi}{31} \sin \frac{2\pi}{31} \sin \frac{4\pi}{31} \sin \frac{8\pi}{31} \cdot 2 \sin \frac{8\pi}{31} \cos \frac{8\pi}{31} \\ &\geq 4^5 \frac{(\sqrt{6} + \sqrt{2})\pi}{4 \cdot 31} \frac{(\sqrt{6} + \sqrt{2})2\pi}{4 \cdot 31} \frac{4\sqrt{3}\pi}{2 \cdot 31} \frac{8\pi}{2 \cdot 31} \cdot 2 \frac{8\pi}{2 \cdot 31} \frac{1}{2} \\ &= 4^5 \left( \frac{\sqrt{6} + \sqrt{2}}{4} \right)^2 \left( \frac{\pi}{31} \right)^5 \sqrt{3} \cdot 64 \geq 4^5 \frac{2 + \sqrt{3}}{4} \frac{1}{10^5} 64 \sqrt{3} \geq \frac{4^7 (3 + 2\sqrt{3})}{10^5} > 1. \end{aligned}$$

This completes the proof. □

In case  $k = 6$ , a numerical experiment suggests that  $\lambda_6 = 1.1197\dots$ . Thus we need a sharper estimate.

### 3 Critical points in the stable sets

Now we consider the orbit of the critical point  $c_z = z - 1$  of  $q_z$ . Set  $z = z_0 = e^{2\pi it}$  with  $t = \frac{1}{2^k - 1}$ , which is a  $k$ -periodic point of  $p$  and  $z_j = p^j(z)$ . By the argument above, it follows

$$2|1 - z_j| \leq \frac{2^{j+2}\pi}{2^k - 1}. \quad (1)$$

In particular, if  $j \leq k - 4$ , we have

$$2|1 - z_j| \leq \frac{2^{k-2}\pi}{2^k - 1} = \frac{\pi}{4} \frac{2^k}{2^k - 1}. \quad (2)$$

**Theorem 3.1.** *Suppose  $k \geq 7$ . Then there exists  $r < 1$  such that  $|Q_z^k(w)| \leq r|w|$  if  $|w| \leq |c_z|$ . Consequently,  $\{|w| \leq |c_z|\} \subset K_z$  and  $c_z \in K_z$ .*

*proof.* If we set  $w_j = Q_z^j(w)$ , it suffices to show  $|w_k| \leq r|w|$  for  $|w| \leq |c_z|$ . First note that, since  $k \geq 7$ ,

$$|c_z| = |z - 1| \leq \frac{2\pi}{2^k - 1} \leq \frac{2\pi}{127} \leq \frac{1}{20},$$

hence for  $|w| \leq |c_z|$ ,

$$|w_1| = |(w + 2(1 - z))w| \leq (|w| + 2|1 - z|)|w| \leq 3|c_z||w| \leq 3\left(\frac{2\pi}{2^k - 1}\right)^2 \leq \frac{1}{100}. \quad (3)$$

Next we show the following.

**Lemma 3.1.**  $|w_j| \leq |w_1|$  for  $1 \leq j \leq k - 3$ .

*proof.* (induction on  $j$ .) The case  $j = 1$  is trivial. We assume the lemma is true for  $j = n \leq k - 4$ . Then

$$|w_{n+1}| = |w_n + 2(1 - z_n)||w_n| \leq (|w_1| + 2|1 - z_n|)|w_n|.$$

By (2) and (3), we have

$$|w_1| + 2|1 - z_n| \leq \frac{1}{100} + \frac{\pi}{4} \frac{2^k}{2^k - 1} \leq \frac{1}{100} + \frac{\pi}{4} \frac{128}{127} < 1.$$

Hence the case  $j = n + 1$  is true. This completes the proof.  $\square$

We will show the lemma also for  $j = k - 2$ . By (1) and (3), we have, for  $1 \leq j \leq k - 3$ ,

$$|w_j| + 2|1 - z_j| \leq 3\left(\frac{2\pi}{2^k - 1}\right)^2 + \frac{2^{j+2}\pi}{2^k - 1} = \frac{2^{j+2}\pi}{2^k - 1} \left(1 + \frac{6\pi}{2^{j+1}(2^k - 1)}\right).$$

Here, since  $k \geq 7$  and  $j \geq 1$ ,

$$\epsilon_j := \frac{3\pi}{2^j(2^k - 1)} \leq \frac{2\pi}{2^k - 1} \leq \frac{1}{20} =: \epsilon.$$

Thus, for  $1 \leq j \leq k - 3$ ,

$$|w_{j+1}| \leq (1 + \epsilon_j) \frac{2^{j+2}\pi}{2^k - 1} |w_j|.$$

Hence we have

$$\begin{aligned} |w_{k-2}| &\leq \left( \prod_{j=1}^{k-3} (1 + \epsilon_j) \frac{2^{j+2}\pi}{2^k - 1} \right) |w_1| \\ &\leq \left( \frac{(1 + \epsilon)4\pi}{2^k - 1} \right)^{k-3} 2^{(k-2)(k-3)/2} |w_1| \\ &\leq \left( (1 + \epsilon)4\pi \frac{2^{(k-2)/2}}{2^k - 1} \right)^{k-3} |w_1| \\ &\leq \left( (1 + \epsilon)4\pi 2^{-(k+2)/2} \frac{1}{1 - \frac{1}{2^k}} \right)^{k-3} |w_1|. \end{aligned}$$

Here, since

$$\frac{1}{1 - \frac{1}{2^k}} = 1 + \frac{1}{2^k} + \frac{1}{2^{2k}} \frac{1}{1 - \frac{1}{2^k}} \leq 1 + \frac{1}{2^k} + \frac{1}{2^{2k-1}},$$

it follows

$$\frac{1 + \epsilon}{1 - \frac{1}{2^k}} \leq (1 + \frac{1}{20}) \left(1 + \frac{1}{128} + \frac{1}{8192}\right) \leq 1 + \frac{6}{100}.$$

By setting  $\epsilon' = \frac{6}{100}$ , we have

$$|w_{k-2}| \leq \left( (1 + \epsilon')4\pi 2^{-(k+2)/2} \right)^{k-3} |w_1|. \quad (4)$$

Since, for  $k \geq 7$ ,

$$(1 + \epsilon')4\pi 2^{-(k+2)/2} \leq \frac{106 \cdot 4\pi}{100 \cdot 2^{9/2}} = \frac{106 \cdot 4\sqrt{2}\pi}{100 \cdot 32} < \frac{106 \cdot 6}{1000} < \frac{2}{3}, \quad (5)$$

it follows  $|w_{k-2}| \leq |w_1|$ .

Now we will estimate  $w_{k-1}$  and  $w_k$ . Since

$$|w_{k-1}| \leq (|w_1| + 4)|w_{k-2}| \leq (4 + \frac{1}{100})|w_{k-2}|,$$

we have

$$\begin{aligned} |w_k| &\leq (|w_{k-1}| + 4)|w_{k-1}| \\ &\leq \left( (4 + \frac{1}{100})|w_{k-2}| + 4 \right) (4 + \frac{1}{100})|w_{k-2}| \\ &\leq \left( (4 + \frac{1}{100})|w_1| + 4 \right) (4 + \frac{1}{100})|w_{k-2}| \\ &\leq \left( 4 + (4 + \frac{1}{100})\frac{1}{100} \right) (4 + \frac{1}{100})|w_{k-2}| \\ &\leq 17|w_{k-2}|. \end{aligned}$$

By (4) and (3), it follows

$$\begin{aligned} |w_k| &\leq 17 \left( (1 + \epsilon')4\pi 2^{-(k+2)/2} \right)^{k-3} |w_1| \\ &\leq 17 \left( (1 + \epsilon')4\pi 2^{-(k+2)/2} \right)^{k-3} 3|c_z||w|. \end{aligned}$$

By (5), it follows

$$17 \left( (1 + \epsilon')4\pi 2^{-(k+2)/2} \right)^{k-3} 3|c_z| \leq 51 \left( \frac{2}{3} \right)^4 |c_z| \leq \frac{51}{20} \left( \frac{2}{3} \right)^4 = \frac{68}{135}.$$

Thus we can take  $r = \frac{68}{135}$ . This completes the proof.  $\square$

Even if  $c_z \in K_z$ ,  $K_z$  and  $J_z$  are not necessarily connected.

**Theorem 3.2.** *The critical point  $c_z$  is escaping for  $z \in J_p$  with  $\operatorname{Re} z < 0$ .*

*proof.* First we show the following.

**Lemma 3.2.**  $K_z \subset \{|w| \leq 5\}$  for  $z \in J_p$ .

*proof.* If  $z \in J_p$ , then  $|z| = 1$ , hence  $|2(1 - z)| \leq 4$ . Thus

$$|q_z(w)| = |w + 2(1 - z)||w| \geq (|w| - 4)|w|.$$

If  $|w| > 5$ , there exists  $r > 0$  such that  $|w| > 5 + r$  and  $|q_z(w)| > (1 + r)|w|$ . That is,  $w$  is escaping. Thus we conclude  $K_z \subset \{|w| \leq 5\}$ .  $\square$

Now since  $q_z(c_z) = (z - 1)^2 + 2(1 - z)(z - 1) = -(z - 1)^2$ , it follows

$$Q_z^2(c_z) = q_{z^2} \circ q_z(c_z) = (z - 1)^4 - 2(1 - z^2)(z - 1)^2 = (3z + 1)(z - 1)^3.$$

If  $|z| = 1$  and  $\operatorname{Re} z < 0$ , then  $|z - 1| > \sqrt{2}$  and  $|3z + 1| = 3|z + 1/3| \geq 3 \cdot 2/3 = 2$ . Therefore we have

$$|Q_z^2(c_z)| = |3z + 1||z - 1|^3 > 2 \cdot (\sqrt{2})^3 = 4\sqrt{2} > 5.$$

By Lemma 3.2, we get the conclusion.  $\square$

Note that, for any  $z \in J_p \setminus \{1\}$ , there exists  $k$  such that  $\operatorname{Re} p^k(z) < 0$ . By Proposition 2.3 in [J], we have the following.

**Corollary 3.1.**  $J_z$  and  $K_z$  are disconnected for  $z \in J_p \setminus \{1\}$ .

We remark that, for  $z = 1$ ,  $K_z = \{|w| \leq 1\}$  and  $J_z = \{|w| = 1\}$  are connected.

## References

- [J] M. Jonsson: Dynamics of polynomial skew products on  $\mathbb{C}^2$ . Math. Ann. 314 (1999), pp. 403–447.