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ASPECTS OF NON UNI-RULED VARIETIES AND ANOTHER REPRESENTATION OF VARIETIES BY PROJECTIVE SYSTEMS IN PROFINITE GROUPS

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Abstract. In this article we construct a projective system of algebraic fundamental groups associated to a system of sub-varieties of a variety over a sub-p-adic field and investigate morphisms between varieties, deformation of a fibre space. We give a proof of higher dimensional Mordell conjecture over a function field ([No], [Km], [F], [Km]).

Contents

1. Introduction

Let K be a sub-p-adic field. We investigate the category of varieties over K by a projective system of profinite groups using Galois-Mochizuki's theory ([Mch], [GG], [SGA]). We construct the projective system of algebraic fundamental groups associated to a system of sub-varieties of a variety over K. By the functor $\mathcal{S}/X \rightsquigarrow \mathcal{P}/\pi_1(X)$, we have a projective system $\pi(X)$ in the product $\Pi_n \Pi_{x \in X^{(n)}} \Pi([x])$, which we denote by $\lim_{\leftarrow x \in X} \Pi([x])$. We obtain the following theorems in the next section:

Theorem 1. *Let X, Y be varieties over K. The following natural map is bijective:*

$$\mathrm{Mor}_K^{surj}(X,Y) \simeq \mathrm{Hom}_{\pi_1(K)}^{open}(\pi(X), \pi(Y)),$$

considered up to composition with an inner automorphism arising from $\ker(\pi(Y) \to \pi_1(K))$.

Theorem 2. *Let p be a prime number and K a sub-p-adic field. Let S be a variety over K and X, Y S-varieties, respectively. Assume that $f : X \to Y$ be a surjective morphism*

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over S and that X is isomorphic to $X_0 \times_K S$ over S. Furthermore suppose $\mathrm{Aut}_S(Y)$ is a locally algebraic group over $\bar{K}$. Then there exist an etale covering $V \to U$ where U is an open set of S and K-variety Y_0 such that $Y_U \cong Y_0 \times_K U$.

In the third section we give some questions about the absolute Galois groups (cf.[L]). We have the following proposition.

Proposition 1. *Let K be a sub-p-adic field. Let X be a projective variety of general type with dimension n. There exist a projective space $\mathbf{P}^n$ and a generically finite surjective morphism $\pi : X \to \mathbf{P}^n$ such that there exist no projective varieties V such that two dominant rational maps*

$$X \to V \to \mathbf{P}^n$$

factor through $\pi : X \to \mathbf{P}^n$.

In the last section we give a proof of the following theorem:

Theorem 3. *Let $f : X \to C$ be a fibre space with the general generic fibre of general type from a projective smooth variety X onto a curve C over the complex number field. Assume there exists a set of sections of X/C which becomes Zariski dense in X. Then* $\mathrm{var}(X/C) = 0$.

2. Another representation of varieties by projective systems

Let X be a locally noetherian scheme and $\bar{x} : \mathrm{Spec}(\Omega) \to X$ a geometric point with a value in an algebraically closed field Ω. Let $\mathcal{C}$ be the category of finite etale coverings of X and F a functor over $\mathcal{C}$ such that $F(X)$ is the set of geometric points of X above $\bar{x}$. Then F is represented by a pro-object P, which is said to be the universal covering of X at the point $\bar{x}$. The fundamental group of X at $\bar{x}$ is defined to be a topological group $\mathrm{Aut}(F)$ or the dual of $\mathrm{Aut}(P)$, which is denoted by $\pi_1(X, \bar{x})$. It is a profinite group, i.e., a projective limit of finite groups. Let $\mathcal{S}$ be the category of locally noetherian schemes and $\mathcal{P}$ that of profinite groups. We thus have a functor $\pi_1 : \mathcal{S} \rightsquigarrow \mathcal{P}$ such that $\pi_1(X)$ is a profinite group for an object X of $\mathcal{S}$. For a morphism $f : X \to Y$ in $\mathcal{S}$ we have the following diagram :

$$\begin{array}{ccc} \mathcal{S}/X & \rightsquigarrow & \mathcal{P}/\pi_1(X) \\ \downarrow & & \downarrow \\ \mathcal{S}/Y & \rightsquigarrow & \mathcal{P}/\pi_1(Y) \end{array}$$

We restrict ourselves to the category of varieties over a certain fields.

Let p be a prime number. Let K be a sub-field of a finitely generated field extension of $\mathbb{Q}_p$, which is called a sub-p-adic field. We investigate the category of varieties over K by representing it to the category of profinite groups. Let X be a normal variety over K

and Oub(X) the set of open subvarieies of X. Let U, V be members of Oub(X) such that $V \subset U$. Then $\pi_1(V) \to \pi_1(U)$ is surjective. We want to transfer the etale topology to the corresponding profinite groups. For an etale covering $V \to U$ $\pi_1(V) \subset \pi_1(U)$, which is open in usual topology. We construct Grothendieck topology on a profinite group $\pi_1(X)$. This topology is generated by a family of usual open mappings and finite surjections such as $\pi_1(V) \to \pi_1(U)$, respectively. Since the generic point η of $X = \cap_{U \in \mathrm{X}} U$, $\pi_1(\eta, \bar{\eta}) \to \pi_1(U, \bar{\eta})$ is surjective. We denote a groupoid $\{\pi_1(X, a)\}$ for all geometric points a by $\pi_1(X)$. If $f : X \to Y$ is dominant between varieties over K, $\pi_1(f) : \pi_1(X) \to \pi_1(Y)$ such that there exists an induced commutative diagram

$$\begin{array}{ccc} \mathrm{Gal}(/\eta_X) & \xrightarrow[\pi_1(f)]{} & \mathrm{Gal}(/\eta_Y) \\ \downarrow & & \downarrow \\ \pi_1(X) & \xrightarrow[\pi_1(f)]{} & \pi_1(Y) \end{array}$$

Endowing this topology to $\pi_1(X)$, we denote it by $\Pi(X)$. We denote by $X^{(n)}$ the set of points of codimension n in X and by $[x]$ the closure of a point x in X, which is a subvariety of codimension n in X. Consider a product $\Pi_n \Pi_{x \in X^{(n)}} \Pi([x])$. By the functor $\mathcal{S}/X \rightsquigarrow \mathcal{P}/\pi_1(X)$, we have a projective system in the product $\Pi_n \Pi_{x \in X^{(n)}} \Pi([x])$, which we denote by $\lim_{\leftarrow\, x \in X} \Pi([x])$.

Definition 1. *Let X be a variety over K. Define a functor π from the category of varieties over K to that of profinite groupoids:*

$$X \rightsquigarrow \lim_{\leftarrow\, x \in X} \Pi([x])$$

We make use of the following Mochizuki's theorem fundamentally.

Theorem 4. [Mch] *Let p be a prime number. Let K be a subfield of a finitely generated field extension of $\mathbb{Q}_p$. Let L, M be function fields of arbitrary dimension over K. Let $\mathrm{Hom}_{\mathrm{Spec(K)}}(\mathrm{Spec(L)}, \mathrm{Spec(M)})$ be the set of K-morphisms from M to L. Let $\mathrm{Hom}_{\Gamma}^{open}(\Gamma_L, \Gamma_M)$ over Γ_K, considered up to composition with an inner automorphism arising from $\ker(\Gamma_\mathrm{M}, \Gamma_\mathrm{K})$, where Γ_L and Γ_M are the absolute Galois groups of L and M, respectively. Then the natural map $\mathrm{Hom}_K(\mathrm{Spec(L)}, \mathrm{Spec(M)}) \to \mathrm{Hom}_{\Gamma_\mathrm{K}}^{\mathrm{open}}(\Gamma_\mathrm{L}, \Gamma_\mathrm{M})$ is bijective.*

Theorem 5. *Let X, Y be varieties over K. The following natural map is bijective:*

$$\mathrm{Mor}_K^{surj}(X, Y) \simeq \mathrm{Hom}_{\pi_1(K)}^{open}(\pi(X), \pi(Y)),$$

considered up to composition with an inner automorphism arising from $\ker(\pi(Y) \to \pi_1(K))$.

Proof. Take a surjection $f : X \to Y$. Naturally f determines a continuous homomorphism $\pi(X) \to \pi(Y)$ over $\pi_1(K)$. Convesely, take a surjective continuous homomorpism $\phi : \pi(X) \to \pi(Y)$. From Mochizuki's theorem above, every point $x \in X$ maps to a point $y \in Y$, respectively. Thus a surjective morphism from X onto Y is determined. □

We hence have the following

$$\mathrm{Aut}_K(X) \simeq \mathrm{Out}_{\pi_1(K)}(\pi(X))$$

Proposition 2. *Let S be a K-variety and $p : X \to S$, $q : Y \to S$ projective smooth varieties over S. Assume that $f : X \to Y$ is a surjective morphism over S there exists an isomorphism $X \cong X_0 \times_K S$ over K, where X_0 is a projective smooth variety over K. Then there exist an etale covering $V \to U$ where U is an open set of S and K-projective smooth variety Y_0 such that $Y_U \cong Y_0 \times_K U$.*

Proof.

$$\Theta_S \to R^1 p_* \Theta_{X/S} \to R^1 p_* f^* \Theta_{Y/S} \leftarrow R^1 q_* \Theta_{Y/S}$$

The last arrow is injective. Kodaira-Spencer map for X/S is zero on the first left map and so Kodaira-Spencer map for Y/S is also zero. We have another proof using the property that $\mathrm{Aut}_S(Y)$ is a locally algebraic group over $\bar{K}$. □

Theorem 6. *Let p be a prime number and K a sub-p-adic field. Let S be a variety over K and X, Y S-varieties, respectively. Assume that $f : X \to Y$ be a surjective morphism over S and that X is isomorphic to $X_0 \times_K S$ over S. Furthermore suppose* $\mathrm{Aut}_S(Y)$ *is a locally algebraic group over $\bar{K}$. Then there exist an etale covering $V \to U$ where U is an open set of S and K-variety Y_0 such that $Y_U \cong Y_0 \times_K U$.*

Proof. We for simplicity assume S is a spectrum of the function field of S. We give the out-line of the proof. The complete proof is published elsewhere. For a surjective S-morphism $f : X \to Y$, we have an exact sequence

$$1 \to \pi(\bar{X}) \to \pi(X) \to \pi(S) \to 1$$

and

$$1 \to \pi(\bar{Y}) \to \pi(Y) \to \pi(S) \to 1$$

By hypothesis $X_{\bar{K}} \cong (X_0 \times_K S)_{\bar{K}}$. $1 \to \pi(\bar{Y}) \to \pi(Y) \to \pi(S) \to 1$ has a dominant section over $\pi_1(S_{\bar{K}})$. Thus the extension class of $\pi(Y_{\bar{K}})$ factors through $H^1(\pi_1(S_{\bar{K}}), \mathrm{Aut}_{\bar{K}}(Y))$. Since $\mathrm{Aut}_{\bar{K}}(Y))$ is a locally algebraic group, the extension class becomes trivial after base-changing S by a finite cover U. We have the following exact sequence

$$1 \to \mathrm{Hom}(\pi_1(K), \mathrm{Out}(\pi(\bar{Y}))) \to \mathrm{Hom}(\pi(S), \mathrm{Out}(\pi(\bar{Y}))) \to \mathrm{Hom}(\pi(S_{\bar{K}}), \mathrm{Out}(\pi(\bar{Y})))$$

Our extension class of $\pi(Y)$ is in the middle term, the image of which is trivial in the third term. Hence we have a variety Y_0 defined over K such that $\pi(Y_0)$ is the extension class in the first term. Therefore

$$Y_U \cong Y_0 \times_K U$$

□

Corollary 1.
- *Y is a projective S-variety.*
- *Y is a spectrum of the function field of an S-variety of Kodaira dimension ≥ 0.*
- *Y is an S-log-variety a compactification of which is projective.*
- *Y is a spectrum of a semi-local S-algebra of height one with Kodaira dimension ≥ 0.*

Corollary 2. *Let $f^o : X^o \to Y^o$ be a log-fibre space over K, ξ_{X^o/Y^o} the relative log canonical invertible sheaf over X and $f : X \to Y$ a semi-stable compactification of X^o/Y^o. Assume $\kappa(\xi_{X^o/Y^o|_{\bar{\eta}}}) \geq 0$. Then*

$$\max_{m>0} \kappa(\det f_* \xi_{X^o/Y^o}^{\otimes m}) \geq \mathrm{var}(X^o/Y^o)$$

Proposition 3. *Let K be a sub-p-adic field and L a function field over K. Let X and Y be a K-variety and a spectrum of a ring R with dimension one over L such that $\kappa(Y) = \dim Y$. The following map is bijective*

$$\mathrm{Mor}_K^{\mathrm{dom}}(X, Y) \simeq \mathrm{Hom}_{\pi_1(K)}^{\mathrm{open}}(\pi_1(X), \pi_1(\mathrm{Spec}(R))),$$

considered up to composition with an inner automorphism arising from $\ker(\pi_1(Y) \to \pi_1(K))$.

Proof. Apply the addition formula to a fibre space $[Y] \to [\mathrm{Spec}(L)]$ associated to $Y \to \mathrm{Spec}(L) = \{\eta\}$. From the formula, we have $\kappa([Y]) \leq \kappa([Y]_{\bar{\eta}}) + \dim_K[\mathrm{Spec}(L)]$. Hence $Y \to \mathrm{Spec}(L) = \eta$ is a hyperbolic curve over L. We first have

$$\mathrm{Mor}_L^{\mathrm{dom}}(X, Y) \simeq \mathrm{Hom}_{\pi_1(L)}^{\mathrm{open}}(\pi_1(X), \pi_1(\mathrm{Spec}(R))),$$

considered up to composition with an inner automorphism arising from $\ker(\pi_1(Y) \to \pi_1(L))$. By induction,

$$\mathrm{Mor}_K^{\mathrm{dom}}(X, Y) \simeq \mathrm{Hom}_{\pi_1(K)}^{\mathrm{open}}(\pi_1(X), \pi_1(\mathrm{Spec}(R))),$$

considered up to composition with an inner automorphism arising from $\ker(\pi_1(Y) \to \pi_1(K))$. □

We make use of the following Mochizuki's theorem A.

Theorem 7. [Mch] *Let p be a prime number. Let K be a subfield of a finitely generated field extension of $\mathbb{Q}_p$. Let X_K be a smooth pro-variety over K and Y_K a hyperbolic pro-curve over K. Let $\mathrm{Hom}_K^{\mathrm{dom}}(X_K, Y_K)$ be the set of dominant K-morphisms from X_K to Y_K and $\mathrm{Hom}_{\Gamma_K}^{\mathrm{open}}(\Pi_{X_K}, \Pi_{Y_K})$ the set of open continuous group homomorphisms $\Pi_{X_K} \to \Pi_{Y_K}$ over Γ_K, modulo up to inner automorphisms arising from $\Delta_{Y_{\bar{K}}}$. Then the natural map*

$$\mathrm{Hom}_K^{\mathrm{dom}}(X_K, Y_K) \to \mathrm{Hom}_{\Gamma_K}^{\mathrm{open}}(\Pi_{X_K}, \Pi_{Y_K})$$

is bijective.

3. Aspect of non uni-ruled varieties

Proposition 4. *Let K be a sub-p-adic field. Let X be a projective variety of general type with dimension n. There exist a projective space $\mathbf{P}^n$ and a generically finite surjective morphism $\pi : X \to \mathbf{P}^n$ such that there exist no projective varieties V such that two dominant rational maps*

$$X \to V \to \mathbf{P}^n$$

factor through $\pi : X \to \mathbf{P}^n$.

Proof. Let $F(X) = \{V | f : X \to V \texttt{ a dominant rational map over } K, \kappa(V) \geq 0\}$. We denote $\Gamma_X = \pi_1(K(X)) = \mathrm{Gal}(/K(X))$ and by $X_V \to X$ a Galois extension $K(V) \to K(X_V)$, respectively. Hence we have the following exact sequences

$$1 \to \Gamma_{X_V} \to \Gamma_V \to \Gamma_V/\Gamma_{X_V} \to 1$$

and

$$1 \to H^1(P, \Gamma_V) \to H^1(\Gamma_V, \Gamma_V) \to H^1(\Gamma_{X_V}, \Gamma_V) \to H^2(P, \Gamma_V),$$

where $P = \Gamma_V/\Gamma_{X_V}$. The last map above is the following push-out to some variety W in the following diagram

$$\begin{array}{ccc}
1 & & 1 \\
\downarrow & & \downarrow \\
\Gamma_{X_V} & \longrightarrow & \Gamma_X \\
\downarrow & & \downarrow \\
\Gamma_V & \longrightarrow & \Gamma_W \\
\downarrow & & \downarrow \\
P & \longrightarrow & P \\
\downarrow & & \downarrow \\
1 & & 1.
\end{array}$$

We also have the following exact sequences

$$1 \to \Gamma_{X_V} \to \Gamma_X \to \Gamma_X/\Gamma_{X_V} \to 1$$

and

$$1 \to H^1(Q, \Gamma_V) \to H^1(\Gamma_X, \Gamma_V) \to H^1(\Gamma_{X_V}, \Gamma_V) \to H^2(Q, \Gamma_V)$$

, where $Q = \Gamma_X / \Gamma_{X_V}$. We estimate $\mathrm{Mor}_K^{\mathrm{dom}}(X, V) \simeq \mathrm{Hom}_{\Gamma_K}^{\mathrm{open}}(\Gamma_X, \Gamma_V)$.

We have $H^1(\Gamma_V, \Gamma_V) \supset \mathrm{Hom}_{\Gamma_K}^{\mathrm{open}}(\Gamma_V, \Gamma_V)$. Since $\dim Bir(V)^o \leq \dim V = n$, we have $\dim \mathrm{Mor}_K^{\mathrm{dom}}(X, V) \leq n$. Thus $\sup_x \dim_x F(X) \leq n$, where x is any point of represented algebraic stack of $F(X)$. On the other hand, $\dim \mathrm{Aut}_K \mathrm{P}^n = \dim PGL(n+1) = (n+1)^2 - 1$. We therefore obtain a projective space P^n such that $\pi : X \to \mathrm{P}^n$ satisfying the condition that there exist no varieties W such that $K(\mathrm{P}^n) \subset K(W) \subset K(X)$.

□

Definition 2. *Let K be a sub-p-adic field. Let Γ be an absolute Galois group of a function field over K. Gamma is said to uni-ruled if there exists an open subgroup of Γ such that the outer automorphism group $\mathrm{Out}_{\Gamma_K}(\Gamma)$ has a non-trivial linear algebraic subgroup. (*[Mat], [Ko], [MP]*)*

Proposition 5. *Let $\mathbf{P}^1$ be a projective space of dimension one over K. Then there exists a ramified covering $\tau : \mathrm{P}^1 \to \mathrm{P}^1$.*

Proof. Let $K[x]$ be a one dimensional polynomial ring. Take $t^n = x$. We have $\mathrm{Spec}(K[t]) \to \mathrm{Spec}(K[x])$ and we get a ramified covering $\tau : \mathbf{P}^1 \to \mathbf{P}^1$. □

Proposition 6. *Let X be a uni-ruled variety. Then there exists a ramified cover $\tau : Y \to X$ such that Y is a uni-ruled variety. In other word, there exist no uni-ruled variety which is generically finite over a non uni-ruled variety.*

Proof. It is easily proven from precedent proposition. □

Let $\bar{K}$ be an algebraic closure of K and $\mathbf{P}_K^1$ a projective line. We denote $\Gamma_{P_K} = \Gamma_{\mathbf{P}_K^1}$ and $\Gamma_{\bar{P}} = \Gamma_{\mathbf{P}_{\bar{K}}^1}$, respectively.

Proposition 7. *Let K be a sub-p-adic field. Let $\mathbf{P}^n$ be the projective space over $\mathbf{Q}$ and $\mathbf{P}_K^n$ the pull-back over K. Let $\Gamma_{P_K^n}$ be the absolute Galois group of the function field of $\mathbf{P}_K^n$.*

Let X be a projective variety over K and Γ_X the absolute Galois group. Then

$$\Gamma_{P_K^n} \simeq \Gamma_{P_K} \times_{\Gamma_K} \cdots \times_{\Gamma_K} \Gamma_{P_K}$$

Proof. Since $\mathbf{P}_K^n$ is birationally equivalent to $\mathbf{P}_K^1 \times_K \cdots {}_K \mathbf{P}_K^1$,

$$\Gamma_{P_K^n} \simeq \Gamma_{P_K} \times_{\Gamma_K} \cdots \times_{\Gamma_K} \Gamma_{P_K}$$

□

Proposition 8. *Let K be a sub-p-adic field, S a function field over K and $\bar{S}$ the algebraic closure in an algebraically closed field. Let $\Gamma_{P^n_S}$ be the absolute Galois group of the function field of $\mathbf{P}^n_S$. Let X be a projective variety over S and Γ_X the absolute Galois group. Let $\bar{X} = X_{\bar{S}}$ and $\bar{P}^n = \mathbf{P}^n_{\bar{S}}$. Assume*

- *$f : X \to \Gamma_{P^n_S}$ is a generically finite morphism over S,*
- *$\Gamma_X \subset \Gamma_{P^n_S}$ is a continuous homomorphism associated to $f : X \to \mathbf{P}^n_S$.,*
- *$\Gamma_X \times_{\Gamma_S} \Gamma_{S_{\bar{K}}} \cong \Gamma_{\bar{X}} \times \Gamma_{S_{\bar{K}}}$*
- *$\Gamma_X \subset \Gamma_{P^n_S}$ induces $\Gamma_{\bar{X}} \times \Gamma_{S_{\bar{K}}} \subset \Gamma_{P^n_{\bar{S}}} \times \Gamma_{S_{\bar{K}}}$ is trivial.*

Then there exists a variety F over K such that

- *$X \cong F \times_K S$*
- *there exists $\Gamma_F \subset \Gamma_{P^n_K}$ over Γ_K such that the base-change by $\Gamma_S \to \Gamma_K$ of the monomorphism above is the original monomorphism $\Gamma_X \subset \Gamma_{P^n_S}$.*

Proof. From the left exact functors $\mathrm{Hom}_{\Gamma_K}(-, \mathrm{Out}_{\Gamma_K}(\Gamma_X))$, $\mathrm{Hom}_{\Gamma_K}(-, \mathrm{Out}_{\Gamma_K}(\Gamma_{P^n_K}))$, we obtain the proof chasing the following diagram of extensions;

$$\begin{array}{ccc} \Gamma_{\bar{X}} & \longrightarrow & \Gamma_{\bar{P}^n} \\ \downarrow & & \downarrow \\ \Gamma_F & \longrightarrow & \Gamma_{P^n} \\ \downarrow & & \downarrow \\ \Gamma_K & \longrightarrow & \Gamma_K \end{array} \qquad \begin{array}{ccc} \Gamma_{\bar{X}} & \longrightarrow & \Gamma_{\bar{P}^n} \\ \downarrow & & \downarrow \\ \Gamma_X & \longrightarrow & \Gamma_{P^n_S} \\ \downarrow & & \downarrow \\ \Gamma_S & \longrightarrow & \Gamma_S \end{array} \qquad \begin{array}{ccc} \Gamma_{\bar{X}} & \longrightarrow & \Gamma_{\bar{P}^n} \\ \downarrow & & \downarrow \\ \Gamma_{\bar{X}} \times \Gamma_{S_{\bar{K}}} & \longrightarrow & \Gamma_{P^n_{\bar{S}}} \times \Gamma_{S_{\bar{K}}} \\ \downarrow & & \downarrow \\ \Gamma_{S_{\bar{K}}} & \longrightarrow & \Gamma_{S_{\bar{K}}} \end{array}$$

□

Definition 3. *Let K be a sub-p-adic field. Let Γ_X be the absolute Galois group of the function field of a variety X over K. Γ_X is said to be uni-rational if there exists an open subgroup Γ_U of Γ_X such that $\Gamma_U \simeq \Gamma_{P_K} \times_{\Gamma_K} \cdots \times_{\Gamma_K} \Gamma_{P_K}$. (cf.*[Ko]*)*

Question 1.

- *If for each open subgroup Γ_U of Γ_X the connected component containing the identity of $\mathrm{Out}(X)$ consists of just one element, then is X of general type?*
- *If there exists an open subgroup Γ_U such that $\mathrm{Out}(\Gamma_U)$ is an abelian variety of maximal dimension, then is $\kappa(X) = 0$ and is the converse valid? (cf.*[Mat], [Mats], [Kaw], [I]*)*

4. Boundedness of sections

We shall show the following

Theorem 8. *Let $f : X \to C$ be a fibre space with the general generic fibre of general type from a projective smooth variety X onto a curve C over the complex number field.*

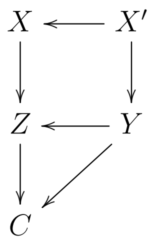
Assume there exists a set of sections of X/C which becomes Zariski dense in X. Then $\mathrm{var}(X/C) = 0$.

Proof. We refer to the case that X is a projective smooth surface S with a canonical divisor K_S. Let $f : S \to C$ be a fibre space with a general fibre of genus $g \geq 2$. Assume S has no (-1)-curves contained in a fibre S/C. Let M be the function field of C and $\bar{M}$ the algebraic closure. Let P be an algebraic point of $S(\bar{M})$ and E_P the curve on S associated to P which is surjective onto C. We denote by E_P^{nor} the normalization of E_P. The geometric canonical height $h_K(P)$ is $h_K(P) = \dfrac{K_{S/C} \cdot E_P}{[M(P) : M]}$. The geometric logarithmic discriminant $d(P)$ is $d(P) = \dfrac{2g(E_P^{nor}) - 2}{[M(P) : M]}$.
Szpiro and Esnauld-Viehweg proved the following estimates of the geometric canonical height $h_K(P)$ ([Szp], [EV], [MP], [Mo1], [Mo2]), respectively;

$$h_K(P) \leq 8 \times 3^{3g+1}(g-1)^2(d(P)/3^g + s + 1 + 1/3^{3g}$$

$$h_K(P) \leq 2(2g-1)^2(d(P) + s).$$

Here s is the number of singular fibres of S/C. The problem is birational. We hence have another fibre space $f : X \to C$ which is birationally equivalent to the original one. $f : X \to C$ factors through $X \to Z$ where Z is a projective smooth variety of $\dim X - 1$. By the formula $\kappa(X) \leq \kappa(X_{\bar{\eta}}) + \dim Z$, $\kappa(X_{\bar{\eta}}) = 1$. Let Y be a ramified cover of Z and projective smooth which is of general type. Take a pull-back of X along $Y \to Z$ and denote by X' a semi-stable reduction of the pull-back.



We shall show $(K_X \cdot E_{P_\lambda}) \leq N_X$ for some number N_X and for any point $P_\lambda \in U(M)$ of bounded degree where U is open in X. Let P_λ be a point of $X(\bar{M})$. Let Q_λ be a point of X' over a point P_λ of X, $Q_\lambda|Y$ the image of Q_λ of X' and $P_\lambda|Z$ the image of P_λ of X, respectively. We claim that there exists a number N_X such that $K_X \cdot E_{P_\lambda} \leq N_X[M(P_\lambda) : M]$. By assumption of recurrence for dimension there exists a number such that $(E_{Q_\lambda|Y} \cdot K_Y) \leq N_Y[M(Q_\lambda|Y) : M]$ for any point $Q_\lambda|Y \in V(\bar{M})$ where V is an open sub-variety of Y. We have a hyper-surface B of Y such that X'/Y has smooth fibres of genus g over $Y \setminus B$. Since Y is of general type we find a number m such that $B \leq mK_Y$. For any curve $E_{Q_\lambda|Y}$ which is not contained in $mK - B$, we have $E_{Q_\lambda|Y} \cdot B \leq E_{Q_\lambda|Y} \cdot mK_Y \leq mN_Y \cdot [M(Q_\lambda|Y) : M]$. If we pull back X'/Y along $E_{Q_\lambda|Y} \subset Y$, we get a surface over a curve $E_{Q_\lambda|Y}$,i.e., $X'/Y|E_{Q_\lambda|Y}$. We apply Esnault-Viehweg's lemma to these fibre spaces. Hence

$$\frac{(X'/Y|E_{Q_\lambda|Y}) \cdot E_{Q_\lambda|Y}}{[M(Q_\lambda) : M]} \leq 2(2g-1)^2(d(P)+s).$$

Here $s \leq mN_Y \cdot [M(Q_{\lambda|Y}) : M]$. For almost all curves E_{P_λ} we have $E_{P_\lambda} \cdot K_X/[M(P_\lambda) : M] \leq E_{Q_\lambda} \cdot K_{X'}/[M(Q_\lambda) : M]$. Since it is enough to consider all the points of bounded degree, we get $(K_X \cdot E_{P_\lambda})/[M(P_\lambda) : M] \leq N_X$ for some number N_X and for any point $P_\lambda \in U(\bar{M})$ where U is open in X. Hence we obtain the set of sections of X/C which is dense in X and which is bounded. We already know that $\mathrm{var}(X/C) = 0$ from [Km]. □

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