ACADEMIC REPORTS Fac. Eng. Tokyo Polytech. Univ. Vol. 33 No.1 (2010)

On numerical types of plain algeblaic curves with the invariant ω

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1 Preliminary

Let C be a curve on a non-singular rational surface W. We start with recalling some definitions on the birational geometry of pairs (W, C).

Definition 1.1 (birational equivalence between pairs)
Let C_1 and C_2 be two curves on non-singular rational surfaces W_1 and W_2, respectively. Suppose that there exists a birational transformation $h : W_1 \to W_2$. We say that h is a birational transformation from (W_1, C_1) to (W_2, C_2), if the proper image $h[C_1]$ of C_1 by h coincides with C_2. We say that the pairs (W_1, C_1) and (W_2, C_2) are birationally equivalent when there exists a birational transformation $h : (W_1, C_1) \to (W_2, C_2)$. In this case, we write $(W_1, C_1) \sim (W_2, C_2)$.

Definition 1.2 (relatively minimal pairs)
A non-singular pair (S, D) is said to be relatively minimal when every exceptional curve E of the first kind on S with $E \neq D$ satisfies $E \quad D \geq 2$.

It is known that every pair (W, C) has a resolution to a non-singular pair (S, D). So we have much interest in non-singular pairs (S, D) which is relatively minimal.

Definition 1.3 (Hirzebruch surface Σ_b)
We define surfaces Σ_b as follows for $b \geq 0$:

$$\Sigma_b = \{((x_0 : x_1 : x_2), (y_0 : y_1)) | x_1 y_1^b = x_2 y_0^b\} \subset \mathbb{P}^2 \quad \mathbb{P}^1.$$

These surfaces are non-singular rational surfaces, which we call Hirzebruch surfaces.

It is well known that every curve C on $\mathbb{P}^2$ is linearly equivalent to dL, where L is a projective straight line on $\mathbb{P}^2$, and d is the degree of C.

It is also well known that every curve C on Σ_b is linearly equivalent to $\sigma \quad _\infty + eF_c$, where $\quad _\infty$ is a section of which the self intersection number is $-b$ and $\rho^{-1}(c)$ is a fiber, for σ and $e \geq 0$. Here, c is a point on $\mathbb{P}^1$ and ρ is the natural projection from Σ_b to $\mathbb{P}^1$.

In that follows, we describe the numerical types of curves C on $\mathbb{P}^2$ as $[d; \nu_0, \nu_1, \ldots, \nu_r]$, where $\nu_0 \geq \nu_1 \geq \ldots \geq \nu_r$ are multiplicities of singular-points $P_0, P_1, \ldots, P_r$ on C including infinitly near singular points.

Similarly, the numerical types of curves C on Σ_b would be described as $[\sigma * e, b; \nu_1, \nu_2, \ldots, \nu_r]$.

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Received Sept.17, 2010

Definition 1.4 (#minimal model)
The pair (Σ_b, C) is said to be #minimal when $\sigma \geq 2\nu_1$ and following conditions are satisfied:

(1) $e \geq \sigma$, when $b = 0$,

(2) $e - \sigma \geq \nu_1$, when $b = 1$ and $r > 0$,

(3) $e - \sigma \geq 2$, when $b = 1$ and $r = 0$.

Theorem 1.5 (Iitaka)
If $\kappa[D] \geq 0$, every non-singular pair (S, D) has a birational regular map to either $(\mathbb{P}^2, D')$ or a minimal resolution of a #minimal(Σ_b, C), where D' is a non-singular plane curve.

2 The invariant ω

The adjunction formula $D \quad (D + K) = 2g - 2.$ indicates that $D + Z$ is closely related to g. Note that g is important but g expresses only the character of curves. So we need an invariant which embodies the properties of the pair (S, D). We recall the old theorem.

Theorem 2.1 (Coolidge)
D is a rational curve and $|D + 2K| = \emptyset \Rightarrow (S, C) \sim (\mathbb{P}^2, L)$

Next, we investigate the property of curves when $|D+2K| \neq \emptyset$. We list several basic results.

Theorem 2.2 (Iitaka2)
If $\sigma \geq 4$, then $D + 2K$ is nef, where the pair (S, D) is minimal.

By the Theorem above, we have $D \quad (D+2K) \geq 0$. Defining α to be $D \quad (D+2K) = 4g-4-D^2$, O.Matsuda[3] succeeded in enumerating the possible numerical types of #minimal models with $\alpha \leq 6$.
The purpose of this paper is to introduce another invariant ω which is related to $D + 3K$ and to enumerate the possible types with small ω.

Definition 2.3
Define ω to be $\frac{1}{2}(D + 3K) \quad D$. Then ω is $3g - 3 - D^2$.

3 Calculating ω of $(\mathbb{P}^2, D)$

By the Theorem 1.5, we know that every non-singular pair has a birational regular map to either a non-singular pair $(\mathbb{P}^2, D)$ or a minimal resolution of a #minimal (Σ_b, C). First, we investigate the relation between ω and $(\mathbb{P}^2, D)$.

In the cace of $(\mathbb{P}^2, D)$, $D \sim dL$ and $K \sim -3L$, where d is the degree of the curve D. We have $D + 3K = (d - 9)L$ and

$$2\omega = D \quad (D + 3K) = d(d - 9).$$

Therefore, minimum value of ω is attained at $d = 4$ or 5.

When $d = 2$, $\kappa[D] = -\infty$. When $d = 3$, the curve D is an elliptic curve and $\omega = -9$. When $d = 4$ or 5, $\omega = -10$ and this is the minimum ω. Furthermore, we can calculate ω as follows:

$$\begin{aligned} d = 6 &\Leftrightarrow \omega = -9, \\ d = 7 &\Leftrightarrow \omega = -7, \\ d = 8 &\Leftrightarrow \omega = -4, \\ d = 9 &\Leftrightarrow \omega = 0, \\ d = 10 &\Leftrightarrow \omega = 5, \\ d = 11 &\Leftrightarrow \omega = 11, \end{aligned}$$

4 #minimal models of (Σ_b, C) with small ω

Next, we investigate the properties of ω for #minimal models (Σ_b, C). In order to determine pairs which have the smallest ω as the starting point of enumerating the types of #minimal models, we recall a theorem concerning $D + 3K$.

Theorem 4.1 (Iitaka)
When the pair (S, D) is minimal and $\sigma \geq 6$, $|D + 3K| \neq \emptyset$, except for the types $[6 * 8, 1; 2^{t_2}]$, where $t_2 \geq 0$.

In that follows, we assume that (S, D) is minimal. Next proposition is well known.

Proposition 4.2
If $\sigma = 2$, then $\kappa[D] = 0, 1$.

In the case when $\kappa[D] = 0, 1$, the types of the pairs have already been enumerated. Hence, in that follows, we assume that $\kappa[D] = 2$.

Proposition 4.3
If $\omega < 0$, then $\nu_1 \leq 2$.

(proof) First, we suppose $\nu_1 \geq 3$, for $\sigma \geq 2\nu_1$ by #minimality. By Theorem 4.1, if $\sigma \geq 6$, there exists an effective divisor Γ which is linearly equivalent to $D + 3K$ except for the types $[6*8, 1; 2^{t_2}]$. But this is the exceptional case, because the pairs have only double points. Suppose that $2\omega = (D+3K)\ \ D < 0$. Since $|D+3K| \neq \emptyset$, we take a member Γ, which satisfies $\Gamma\ \ D < 0$. Then Γ is written as $\Gamma' + \alpha D (\alpha > 0)$, where Γ' is another effective divisor. It follows that $\Gamma - D$ is effective. However, $\Gamma - D$ is linearly equivalent to $3K$, which induces that $|3K| \neq \emptyset$. This contradicts the fact that the Kodaira dimension of a rational surface is $-\infty$. This completes the proof.

If the type of the pairs is $[\sigma * e, b; \nu_1, \nu_2, \ldots, \nu_r]$, then we have

$$D^2 = 2\sigma e - b\sigma^2 - \sum_{j=1}^{r} \nu_j^2.$$

and

$$g = (e-1)(\sigma-1) - \frac{\sigma(\sigma-1)b}{2} - \sum_{j=1}^{r} \frac{\nu_j(\nu_j-1)}{2}.$$

Let m be the maximal multiplicty of singular points on the curve C. Let t_i denote the number of the singular points which have the multiplicity i. By Proposition 4.3, when $\omega < 0$, we obtain

$$\begin{aligned} D^2 &= 2\sigma e - b\sigma^2 - 4t_2. \\ g &= (e-1)(\sigma-1) - \frac{\sigma(\sigma-1)b}{2} - t_2. \end{aligned}$$

Then,

$$\begin{aligned} \omega &= 3g - 3 - D^2 \\ &= 3(e-1)(\sigma-1) - \frac{3}{2}\sigma(\sigma-1)b - 3 - 2\sigma e + b\sigma^2 + 4t_2 \\ &= e\sigma - 3e - 3\sigma - \frac{1}{2}b\sigma^2 + \frac{3}{2}b\sigma + t_2. \end{aligned}$$

Therefore,

$$(\sigma-3)(2e - b\sigma - 6) + 2t_2 = 2(\omega+9).$$

Lemma 4.4
If $\kappa[D] = 2$, then $2e - b\sigma - 6 \geq 0$.

(proof)For $\kappa[D] = 2$, we have $\sigma \geq 3$ by Proposition 4.2.
1.In the case when $b = 0$, we have $2e - b\sigma - 6 = 2e - 6$. By $\sigma \geq 3$ and #minimality condition $e \geq \sigma$, Lemma holds.
2.In the case when $b = 1$, we have $2e - b\sigma - 6 = 2e - \sigma - 6$. By $\sigma \geq 3$ and #minimality condition $e - \sigma > 1$,

$$2e - \sigma - 6 \geq 2(\sigma+2) - \sigma - 6 = \sigma - 2 > 0.$$

3.In the case when $b \geq 2$, rewriting $2e - b\sigma - 6$ as $(e - b\sigma) + (e - 6)$, we have the inequality by the known property $e - b\sigma \geq 0$ and the condition $\sigma \geq 3$.

Proposition 4.5
If $\kappa[D] = 2$ and $\omega < 0$, then

$$(\sigma-3)(2e - b\sigma - 6) + 2t_2 = 2(\omega+9) \geq 0.$$

Therefore, $\omega \geq -9$.

(proof)By Proposition 4.2, the Lemma above and Proposition 4.3, the inequality has been established.

In that follows, we shall enumerate the possible numerical types of pairs which have small ω, i.e. $\omega = -9, -8, \ldots$.

5 Case $\omega = -9$

By Proposition 4.5, if $\omega = -9$, then $t_2 = 0$.

1.In the case $b = 0$, we have $2e - b\sigma - 6 = 2e - 6$. By $\sigma \geq 3$ and $e \geq \sigma$, if $\sigma \geq 4$, e is also gleater than 3 and $2e - 6 \neq 0$. So $\sigma = 3$ and the possible types are

$$[3 * e, 0; 1], \text{ where } \ e = 3 + u \ (u \geq 0).$$

2.In the case $b = 1$, we have $2e - b\sigma - 6 = 2e - \sigma - 6$. By $\sigma \geq 3$ and #minimality condition $e - \sigma > 1$, if $\sigma \geq 4$, e must be gleater than 6. But positive integer solution of $2e - \sigma - 6 = 0$ satisfying $e - \sigma > 1$ does not exist. Hence $\sigma = 3$ and the possible types are

$$[3 * e, 1; 1], \text{ where } \ e = 5 + u \ (u \geq 0).$$

3.In the case $b \geq 2$, we rewrite $2e - b\sigma - 6$ as $(e - b\sigma) + (e - 6)$. By the known condition $e - b\sigma \geq 0$, if $\sigma \geq 4$, then $(e - 6) \geq (b\sigma - 6) \geq (2 \quad 4 - 6) > 0$. Therefore, $\sigma = 3$ and the possible types are

$$[3 * e, b; 1], \text{ where } \ e = 3b + u \ (u \geq 0, b \geq 2).$$

6 Case $\omega = -8$

By Proposition 4.5, we have

$$(\sigma - 3)(2e - b\sigma - 6) + 2t_2 = 2(\omega + 9) = 2.$$

By Proposition 4.2 and Lemma 4.4, we have $t_2 = 0, 1$. First, we prepare another Lemma.

Lemma 6.1
If $\kappa[D] = 2$ and $\sigma \geq 4$, then

$$\sigma - 3 < 2e - b\sigma - 6.$$

(proof)1. In the case when $b = 0$,

$$2e - b\sigma - 6 - (\sigma - 3) = 2e - \sigma - 3 \geq 2\sigma - \sigma - 3 = \sigma - 3 > 0.$$

2. In the case when $b = 1$, by the #minimality condition $e - \sigma \geq 2$,

$$2e - b\sigma - 6 - (\sigma - 3) = 2e - 2\sigma - 3 > 0.$$

3. In the case when $b \geq 2$, by the condition $e - b\sigma \geq 0$ and $\sigma \geq 4$, we have $e - \sigma \geq 4$. Then

$$2e - b\sigma - 6 - (\sigma - 3) = e - b\sigma + e - \sigma - 3 > 0.$$

6.1 Case $t_2 = 0$

1. In the case when $b = 0$, we have $(\sigma - 3)(e - 3) = 1$. By the possible factorization of the right side and the assumption of $\kappa[D] \geq 2$, i.e. ,$\sigma \geq 3$, the only possible type is

$$[4 * 4, 0; 1].$$

2. In the case when $b = 1$, we have $(\sigma - 3)(2e - \sigma - 6) = 2$. By the same factorization above and Lemma 6.1, we have $\sigma = 4$. If $\sigma = 4$, then $e = 6$. So the possible type is

$$[4 * 6, 1; 1].$$

3. In the case when $b \geq 2$, we have $(\sigma - 3)(2e - b\sigma - 6) = 2$. By the same factorization above and Lemma 6.1, we have $\sigma = 4$. It follows that $2e - 4b = 8$. By the condition $e - b\sigma \geq 0$, i.e. $e \geq 4b$, the only possible solution of b is 2 and hence $e = 8$. So the type becomes

$$[4 * 8, 2; 1].$$

6.2 Case $t_2 = 1$

We have $(\sigma - 3)(2e - b\sigma - 6) = 0$. But, by the #minimality condition $\sigma \geq 2\nu_1 = 4$ and Lemma6.1, there exist no solution of σ, e. This implies that when $t_2 = \omega + 9$, there exist no #minimal model. So we obtain the following

Proposition 6.2
If $\kappa[D] = 2$ and $-8 \leq \omega < 0$, then $t_2 \leq \omega + 8$.

7 Case $\omega = -7$

By Proposition 4.5, we have

$$(\sigma - 3)(2e - b\sigma - 6) + 2t_2 = 4.$$

By Proposition 4.2, Lemma4.4 and Lemma6.1, we have $t_2 = 0, 2$.

7.1 Case $t_2 = 0$

1. In the case when $b = 0$, we have $(\sigma - 3)(e - 3) = 2$. By the possible factorization of the right hand side, the assumption of $\kappa[D] \geq 2$ i.e. $\sigma \geq 3$, and #minimality condition $e \geq \sigma$, the only possible type is

$$[4 * 5, 0; 1].$$

2. In the case when $b = 1$, we have $(\sigma - 3)(2e - \sigma - 6) = 4$. By the same factorization above and Lemma 6.1, we have $\sigma = 4$. If $\sigma = 4$, then $e = 7$. So the possible type is

$$[4 * 7, 1; 1].$$

3. In the case when $b \geq 2$, we have $(\sigma - 3)(2e - b\sigma - 6) = 4$. By the same factorization above and Lemma 6.1, we have $\sigma = 4$. It follows that $2e - 4b = 10$. By the condition $e - b\sigma \geq 0$, i.e. $e \geq 4b$, the only possible solution of b is 2 and it follows that $e = 9$. So the type is

$$[4 * 9, 2; 1].$$

7.2 Case $t_2 = 1$

We have

$$(\sigma - 3)(2e - b\sigma - 6) = 2.$$

Noting that #minimality condition in the case $b = 1$ with only double points is the same as the case when C is non-singular, we can use the same solution of σ, e and b in the case of section 6.1. So the possible types are

$$[4 * 4, 0; 2], [4 * 6, 1; 2], [4 * 8, 2; 2].$$

8 Case $\omega = -6$

By Proposition 4.5, we have

$$(\sigma - 3)(2e - b\sigma - 6) + 2t_2 = 6.$$

By Proposition 4.2 and Lemma 4.4, we have $t_2 = 0, 1, 2$.

8.1 Case $t_2 = 0$

1. In the case when $b = 0$, we have $(\sigma - 3)(e - 3) = 3$. By the possible factorization of the right side, the condition $\sigma \geq 3$, and #minimality condition $e \geq \sigma$, the only possible type is

$$[4 * 6, 0; 1].$$

2. In the case when $b = 1$, we have $(\sigma - 3)(2e - \sigma - 6) = 6$. By the same factorization above and Lemma 6.1, we have $\sigma = 4$ or 5. If $\sigma = 4$, then $e = 8$. If $\sigma = 5$, then $e = 7$. So the possible types are

$$[4 * 8, 1; 1], [5 * 7, 1; 1].$$

3. In the case when $b \geq 2$, we have $(\sigma - 3)(2e - b\sigma - 6) = 6$. By the same factorization above and Lemma 6.1, we have $\sigma = 4, 5$. If $\sigma = 4$, then $2e - 4b = 12$. By the condition $e - b\sigma \geq 0$,

i.e. $e \geq 4b$, the possible solutions of b are $2, 3$ and it follows that $e = 10, 12$, respectively. If $\sigma = 5$, then $2e - 5b = 9$. There exist no solutions of b under the condition $e - b\sigma \geq 0$, i.e. , $e \geq 5b$. So the types are

$$[4 * 10, 2; 1], [4 * 12, 3; 1].$$

8.2 Case $t_2 = 1$

We have

$$(\sigma - 3)(2e - b\sigma - 6) = 4.$$

We can use the same solution of σ, e and b in the case of section 7.1. So the possible types are

$$[4 * 5, 0; 2], [4 * 7, 1; 2], [4 * 9, 2; 2].$$

8.3 Case $t_2 = 2$

We have

$$(\sigma - 3)(2e - b\sigma - 6) = 2.$$

We can use the same solution of σ, e and b in the case of section 6.1. So the possible types are

$$[4 * 4, 0; 2^2], [4 * 6, 1; 2^2], [4 * 8, 2; 2^2].$$

9 Case $\omega = -5$

By Proposition 4.5, we have

$$(\sigma - 3)(2e - b\sigma - 6) + 2t_2 = 8.$$

By Proposition 4.2 and Lemma 4.4, we have $t_2 = 0, 1, 2, 3$.

9.1 Case $t_2 = 0$

1. In the case when $b = 0$, we have $(\sigma - 3)(e - 3) = 4$. By the possible factorization of the right side, the condition $\sigma \geq 3$, and #minimality condition $e \geq \sigma$, we have $\sigma = 4, 5$ and it follows that $e = 7, 5$, respectively. So the possible types are

$$[4 * 7, 0; 1], [5 * 5, 0; 1].$$

2. In the case when $b = 1$, we have $(\sigma - 3)(2e - \sigma - 6) = 8$. By the same factorization above and Lemma 6.1, we have $\sigma = 4$ or 5. If $\sigma = 4$, then $e = 9$. But if $\sigma = 5$, then e is not an integer. So the only possible type is

$$[4 * 9, 1; 1].$$

3. In the case when $b \geq 2$, we have $(\sigma - 3)(2e - b\sigma - 6) = 8$. By the same factorization above and Lemma 6.1, we have $\sigma = 4, 5$. If $\sigma = 4$, then $2e - 4b = 14$. By the condition $e - b\sigma \geq 0$, i.e. $e \geq 4b$, the possible solutions of b are $2, 3$ and it follows that $e = 11, 13$, respectively. If $\sigma = 5$, then $2e - 5b = 10$. By the condition $e \geq 5b$, the only solutions of b is 2 and it follows that $e = 10$. So the types are

$$[4 * 11, 2; 1], [4 * 13, 3; 1], [5 * 10, 2; 1].$$

9.2 Case $t_2 = 1$

We have

$$(\sigma - 3)(2e - b\sigma - 6) = 6.$$

We can use the same solution of σ, e and b in the case of section 8.1. So the possible types are

$$[4 * 6, 0; 2], [4 * 8, 1; 2], [5 * 7, 1; 2], [4 * 10, 2; 2], [4 * 12, 3; 2].$$

9.3 Case $t_2 = 2$

We have

$$(\sigma - 3)(2e - b\sigma - 6) = 4.$$

We can use the same solution of σ, e and b in the case of section 7.1. So the possible types are

$$[4 * 5, 0; 2^2], [4 * 7, 1; 2^2], [4 * 9, 2; 2^2].$$

9.4 Case $t_2 = 3$

We have

$$(\sigma - 3)(2e - b\sigma - 6) = 2.$$

We can use the same solution of σ, e and b in the case of section 6.1. So the possible types are

$$[4 * 4, 0; 2^3], [4 * 6, 1; 2^3], [4 * 8, 2; 2^3].$$

10 Case $\omega = -4$

By Proposition 4.5, we have

$$(\sigma - 3)(2e - b\sigma - 6) + 2t_2 = 10.$$

By Proposition 4.2 and Lemma 4.4, we have $t_2 = 0, 1, 2, 3, 4$.

10.1 Case $t_2 = 0$

1. In the case when $b = 0$, we have $(\sigma - 3)(e - 3) = 5$. By the possible factorization of the right side, the condition $\sigma \geq 3$, and #minimality condition $e \geq \sigma$, we have $\sigma = 4$ and it follows that $e = 8$. So the only possible type is

$$[4 * 8, 0; 1].$$

2. In the case when $b = 1$, we have $(\sigma - 3)(2e - \sigma - 6) = 10$. By the same factorization above and Lemma 6.1, we have $\sigma = 4, 5$ and it follows that $e = 10, 8$, respectively. So the only possible types are

$$[4 * 10, 1; 1], [5 * 8, 1; 1].$$

3. In the case when $b \geq 2$, we have $(\sigma - 3)(2e - b\sigma - 6) = 10$. By the same factorization above and Lemma 6.1, we have $\sigma = 4, 5$. If $\sigma = 4$, then $2e - 4b = 16$. By the condition $e - b\sigma \geq 0$, i.e. $e \geq 4b$, the possible solutions of b are $2, 3, 4$ and it follows that $e = 12, 14, 16$, respectively. If $\sigma = 5$, then $2e - 5b = 9$. In this case, there exists no solutions of b under the condition $e - b\sigma \geq 0$ i.e. $e \geq 5b$. So the types are

$$[4 * 12, 2; 1], [4 * 14, 3; 1], [4 * 16, 4; 1].$$

10.2 Case $t_2 = 1$

We have

$$(\sigma - 3)(2e - b\sigma - 6) = 8.$$

We can use the same solution of σ, e and b in the case of section 9.1. So the possible types are

$$[4 * 7, 0; 2], [5 * 5, 0; 2], [4 * 9, 1; 2], [4 * 11, 2; 2], [4 * 13, 3; 2], [5 * 10, 2; 2].$$

10.3 Case $t_2 = 2$

We have

$$(\sigma - 3)(2e - b\sigma - 6) = 6.$$

We can use the same solution of σ, e and b in the case of section 8.1. So the possible types are

$$[4 * 6, 0; 2^2], [4 * 8, 1; 2^2], [5 * 7, 1; 2^2], [4 * 10, 2; 2^2], [4 * 2, 3; 2^2].$$

10.4 Case $t_2 = 3$

We have

$$(\sigma - 3)(2e - b\sigma - 6) = 4.$$

We can use the same solution of σ, e and b in the case of section 7.1. So the possible types are

$$[4 * 5, 0; 2^3], [4 * 7, 1; 2^3], [4 * 9, 2; 2^3].$$

10.5 Case $t_2 = 4$

We have

$$(\sigma - 3)(2e - b\sigma - 6) = 2.$$

We can use the same solution of σ, e and b in the case of section 6.1. So the possible types are

$$[4 * 4, 0; 2^4], [4 * 6, 1; 2^4], [4 * 8, 2; 2^4].$$

11 Case $\omega = -3$

By Proposition 4.5, we have

$$(\sigma - 3)(2e - b\sigma - 6) + 2t_2 = 12.$$

By Proposition 4.2 and Lemma 4.4, we have $t_2 = 0, 1, 2, 3, 4, 5$.

11.1 Case $t_2 = 0$

1. In the case when $b = 0$, we have $(\sigma - 3)(e - 3) = 6$. By the possible factorization of the right side, the condition $\sigma \geq 3$, and #minimality condition $e \geq \sigma$, we have $\sigma = 4, 5$ and it follows that $e = 9, 6$, respectively. So the possible types are

$$[4 * 9, 0; 1], [5 * 6, 0; 1].$$

2. In the case when $b = 1$, we have $(\sigma - 3)(2e - \sigma - 6) = 12$. By the same factorization above and Lemma 6.1, we have $\sigma = 4, 6$ and it follows that $e = 11, 8$, respectively. So the possible types are

$$[4 * 11, 1; 1], [6 * 8, 1; 1].$$

3. In the case when $b \geq 2$, we have $(\sigma - 3)(2e - b\sigma - 6) = 12$. By the same factorization above and Lemma 6.1, we have $\sigma = 4, 5, 6$. If $\sigma = 4$, then $2e - 4b = 18$. By the condition $e - b\sigma \geq 0$, i.e. $e \geq 4b$, the possible solutions of b are $2, 3, 4$ and it follows that $e = 13, 15, 17$, respectively. If $\sigma = 5$, then $2e - 5b = 12$. By $e \geq 5b$, the only solution of b is 2 and it follows that $e - 11$. If $\sigma = 6$, then $2e - 6b = 10$. In this case, there exist no solutions of b under the condition $e - b\sigma \geq 0$ i.e. $e \geq 6b$. So the types are

$$[4 * 13, 2; 1], [4 * 15, 3; 1], [4 * 17, 4; 1], [5 * 11, 2; 1].$$

11.2 Case $t_2 = 1$

We have

$$(\sigma - 3)(2e - b\sigma - 6) = 10.$$

We can use the same solution of σ, e and b in the case of section 10.1. So the possible types are

$$[4 * 8, 0; 2], [4 * 10, 1; 2], [5 * 8, 1; 2], [4 * 12, 2; 2], [4 * 14, 3; 2], [4 * 16, 4; 2].$$

11.3 Case $t_2 = 2$

We have

$$(\sigma - 3)(2e - b\sigma - 6) = 8.$$

We can use the same solution of σ, e and b in the case of section 9.1. So the possible types are

$$[4 * 7, 0; 2^2], [5 * 5, 0; 2^2], [4 * 9, 1; 2^2], [4 * 11, 2; 2^2], [4 * 13, 3; 2^2], [5 * 10, 2; 2^2].$$

11.4 Case $t_2 = 3$

We have

$$(\sigma - 3)(2e - b\sigma - 6) = 6.$$

We can use the same solution of σ, e and b in the case of section 8.1. So the possible types are

$$[4 * 6, 0; 2^3], [4 * 8, 1; 2^3], [5 * 7, 1; 2^3], [4 * 10, 2; 2^3], [4 * 2, 3; 2^3].$$

11.5 Case $t_2 = 4$

We have

$$(\sigma - 3)(2e - b\sigma - 6) = 4.$$

We can use the same solution of σ, e and b in the case of section 7.1. So the possible types are

$$[4 * 5, 0; 2^4], [4 * 7, 1; 2^4], [4 * 9, 2; 2^4].$$

11.6 Case $t_2 = 5$

We have

$$(\sigma - 3)(2e - b\sigma - 6) = 2.$$

We can use the same solution of σ, e and b in the case of section 6.1. So the possible types are

$$[4 * 4, 0; 2^5], [4 * 6, 1; 2^5], [4 * 8, 2; 2^5].$$

12 Case $\omega = -2$

By Proposition 4.5, we have

$$(\sigma - 3)(2e - b\sigma - 6) + 2t_2 = 14.$$

By Proposition 4.2 and Lemma 4.4, we have $t_2 = 0, 1, 2, 3, 4, 5, 6$.

12.1 Case $t_2 = 0$

1. In the case when $b = 0$, we have $(\sigma - 3)(e - 3) = 7$. By the possible factorization of the right side, the condition $\sigma \geq 3$, and #minimality condition $e \geq \sigma$, we have $\sigma = 4$ and it follows that $e = 10$. So the only possible type is

$$[4 * 10, 0; 1].$$

2. In the case when $b = 1$, we have $(\sigma - 3)(2e - \sigma - 6) = 14$. By the same factorization above and Lemma 6.1, we have $\sigma = 4, 5$ and it follows that $e = 12, 9$, respectively. So the possible types are

$$[4 * 12, 1; 1], [5 * 9, 1; 1].$$

3. In the case when $b \geq 2$, we have $(\sigma - 3)(2e - b\sigma - 6) = 14$. By the same factorization above and Lemma 6.1, we have $\sigma = 4, 5$. If $\sigma = 4$, then $2e - 4b = 20$. By the condition $e - b\sigma \geq 0$, i.e. $e \geq 4b$, the possible solutions of b are $2, 3, 4, 5$ and it follows that $e = 14, 16, 18, 20$, respectively. If $\sigma = 5$, then $2e - 5b = 13$. Under the condition $e \geq 5b$, there exist no solutions of b in this case. So the types are

$$[4 * 14, 2; 1], [4 * 16, 3; 1], [4 * 18, 4; 1], [4 * 20, 5; 1].$$

12.2 Case $t_2 = 1$

We have

$$(\sigma - 3)(2e - b\sigma - 6) = 12.$$

We can use the same solution of σ, e and b in the case of section 11.1. So the possible types are

$$[4 * 9, 0; 2], [5 * 6, 0; 2], [4 * 11, 1; 2], [6 * 8, 1; 2], [4 * 13, 2; 2], [4 * 15, 3; 2], [4 * 17, 4; 2], [5 * 11, 2; 2].$$

12.3 Case $t_2 = 2$

We have

$$(\sigma - 3)(2e - b\sigma - 6) = 10.$$

We can use the same solution of σ, e and b in the case of section 10.1. So the possible types are

$$[4 * 8, 0; 2^2], [4 * 10, 1; 2^2], [5 * 8, 1; 2^2], [4 * 12, 2; 2^2], [4 * 14, 3; 2^2], [4 * 16, 4; 2^2].$$

12.4 Case $t_2 = 3$

We have

$$(\sigma - 3)(2e - b\sigma - 6) = 8.$$

We can use the same solution of σ, e and b in the case of section 9.1. So the possible types are

$$[4 * 7, 0; 2^3], [5 * 5, 0; 2^3], [4 * 9, 1; 2^3], [4 * 11, 2; 2^3], [4 * 13, 3; 2^3], [5 * 10, 2; 2^3].$$

12.5 Case $t_2 = 4$

We have

$$(\sigma - 3)(2e - b\sigma - 6) = 6.$$

We can use the same solution of σ, e and b in the case of section 8.1. So the possible types are

$$[4 * 6, 0; 2^4], [4 * 8, 1; 2^4], [5 * 7, 1; 2^4], [4 * 10, 2; 2^4], [4 * 2, 3; 2^4].$$

12.6 Case $t_2 = 5$

We have

$$(\sigma - 3)(2e - b\sigma - 6) = 4.$$

We can use the same solution of σ, e and b in the case of section 7.1. So the possible types are

$$[4 * 5, 0; 2^5], [4 * 7, 1; 2^5], [4 * 9, 2; 2^5].$$

12.7 Case $t_2 = 6$

We have

$$(\sigma - 3)(2e - b\sigma - 6) = 2.$$

We can use the same solution of σ, e and b in the case of section 6.1. So the possible types are

$$[4 * 4, 0; 2^6], [4 * 6, 1; 2^6], [4 * 8, 2; 2^6].$$

13 Case $\omega = -1$

By Proposition 4.5, we have

$$(\sigma - 3)(2e - b\sigma - 6) + 2t_2 = 16.$$

By Proposition 4.2 and Lemma 4.4, we have $t_2 = 0, 1, 2, 3, 4, 5, 6, 7$.

13.1 Case $t_2 = 0$

1. In the case when $b = 0$, we have $(\sigma - 3)(e - 3) = 8$. By the possible factorization of the right side, the condition $\sigma \geq 3$, and #minimality condition $e \geq \sigma$, we have $\sigma = 4, 5$ and it follows that $e = 11, 7$, respectively. So the possible types are

$$[4 * 11, 0; 1], [5 * 7, 0; 1].$$

2. In the case when $b = 1$, we have $(\sigma - 3)(2e - \sigma - 6) = 16$. By the same factorization above and Lemma 6.1, we have $\sigma = 4$ and it follows that $e = 13$. So the only possible type is

$$[4 * 13, 1; 1].$$

3. In the case when $b \geq 2$, we have $(\sigma - 3)(2e - b\sigma - 6) = 16$. By the same factorization above and Lemma 6.1, we have $\sigma = 4, 5$. If $\sigma = 4$, then $2e - 4b = 22$. By the condition $e - b\sigma \geq 0$, i.e. $e \geq 4b$, the possible solutions of b are $2, 3, 4$ and it follows that $e = 15, 17, 19$, respectively. If $\sigma = 5$, then $2e - 5b = 14$. By $e \geq 5b$, the only solution of b is 2 and it follows that $e = 12$. So the types are

$$[4 * 15, 2; 1], [4 * 17, 3; 1], [4 * 19, 4; 1], [5 * 12, 2; 1].$$

13.2 Case $t_2 = 1$

We have

$$(\sigma - 3)(2e - b\sigma - 6) = 10.$$

We can use the same solution of σ, e and b in the case of section 12.1. So the possible types are

$$[4 * 10, 0; 2], [4 * 12, 1; 2], [5 * 9, 1; 2], [4 * 14, 2; 2], [4 * 16, 3; 2], [4 * 18, 4; 2], [4 * 20, 5; 2].$$

13.3 Case $t_2 = 2$

We have

$$(\sigma - 3)(2e - b\sigma - 6) = 12.$$

We can use the same solution of σ, e and b in the case of section 11.1. So the possible types are

$$[4*9, 0; 2^2], [5*6, 0; 2^2], [4*11, 1; 2^2], [6*8, 1; 2^2], [4*13, 2; 2^2], [4*15, 3; 2^2], [4*17, 4; 2^2], [5*11, 2; 2^2].$$

13.4 Case $t_2 = 3$

We have

$$(\sigma - 3)(2e - b\sigma - 6) = 10.$$

We can use the same solution of σ, e and b in the case of section 10.1. So the possible types are

$$[4 * 8, 0; 2^3], [4 * 10, 1; 2^3], [5 * 8, 1; 2^3], [4 * 12, 2; 2^3], [4 * 14, 3; 2^3], [4 * 16, 4; 2^3].$$

13.5 Case $t_2 = 4$

We have

$$(\sigma - 3)(2e - b\sigma - 6) = 8.$$

We can use the same solution of σ, e and b in the case of section 9.1. So the possible types are

$$[4 * 7, 0; 2^4], [5 * 5, 0; 2^4], [4 * 9, 1; 2^4], [4 * 11, 2; 2^4], [4 * 13, 3; 2^4], [5 * 10, 2; 2^4].$$

13.6 Case $t_2 = 5$

We have

$$(\sigma - 3)(2e - b\sigma - 6) = 6.$$

We can use the same solution of σ, e and b in the case of section 8.1. So the possible types are

$$[4 * 6, 0; 2^5], [4 * 8, 1; 2^5], [5 * 7, 1; 2^5], [4 * 10, 2; 2^5], [4 * 2, 3; 2^5].$$

13.7 Case $t_2 = 6$

We have

$$(\sigma - 3)(2e - b\sigma - 6) = 4.$$

We can use the same solution of σ, e and b in the case of section 7.1. So the possible types are

$$[4 * 5, 0; 2^6], [4 * 7, 1; 2^6], [4 * 9, 2; 2^6].$$

13.8 Case $t_2 = 7$

We have

$$(\sigma - 3)(2e - b\sigma - 6) = 2.$$

We can use the same solution of σ, e and b in the case of section 6.1. So the possible types are

$$[4 * 4, 0; 2^7], [4 * 6, 1; 2^7], [4 * 8, 2; 2^7].$$

[references]

[1] Shigeru Iitaka, Heimenkyokusen no Kika, Kyouritsu, 2001.

[2] R. Hartshorne, Curves with high self-intersection on algebraic surfaces, Publ. I.H.E.S., (1970), 111-126.

[3] Osamu Matsuda, On numerical types of algebraic curves on rational surfaces, Tokyo J Math 24 (2002), no2, 359-367.