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HIGHER DIMENSIONAL DIOPHANTINE PROBLEMS

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ABSTRACT. In this article we treat an analogue of higher Modell conjecture over function field([No], [Km]). Furthermore we apply a similar argument to the arithmetic case([F], [L]), [Mo1]).

1. INTRODUCTION

We shall prove a weak version of the following conjectures proposed by Lang and Bombieri([L]):

Conjecture 1. *Let K be an arithmetic field and X a variety defined over K. Assume that X be a variety of general type. Then it has no dense set of K-rational points in X.*

When $\dim X = 1$, it is the famous Faltings theorem([F]).

There is an analogue of the conjecture, which is proposed by Noguchi([No], [Km]):

Conjecture 2. *Let X and S be algebraic varieties over the field of the complex numbers. Assume that X/S be a fibre space with the geometric generic fibre of general type. If X/S has a dense set of rational sections in X, then* $\mathrm{var}(X/S) = 0$.

2. GEOMETRIC CASE

Lemma 1. *Let k be a field of characteristic 0 and C a non singular curve over k. Let $f : X \to C$ be a projective surjective morphism between non singular varieties over k with connected fibres and let $\pi : \mathbf{P}(\Omega_X^{\otimes n}) \to X$ be the structure morphism of projective bundle where $n = \dim X$. Then*

(1) *there exists an exact sequence*

$$0 \to f^*\Omega_C \to \Omega_X \to \Omega_{X/C} \to 0$$

(2) *there exists an epimorphism onto the fundamental invertible sheaf.*

$$\pi^*\Omega_X^{\otimes n} \to \mathcal{O}_P(1)$$

(3) *if X is of general type, the fundamental invertible sheaf $\mathcal{O}_P(1)$ is big.*

(4) *if $\Omega_C^{\otimes n}$ is effective, the fundamental invertible sheaf $\mathcal{O}_P(1)$ is effective.*

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Proof. (1) It is well-known.

(2) See EGA1 ([I].

(3) Since there exists an inclusion $\omega_X \subset \Omega_X^{\otimes n}$, we have

$$\pi^*\omega_X \to \pi^*\Omega_X^{\otimes n} \to \mathcal{O}_P(1)$$

and its adjunction

$$\omega_X \to \pi_*\mathcal{O}_P(1) = \Omega_X^{\otimes n}$$

Since ω_X is big and $\mathcal{O}_P(1)$ is π-ample, there exists a number b such that $\omega_X^{\otimes b} \otimes \mathcal{O}_P(1)$ is big. Hence $\mathcal{O}_P(b+1)$ is big. Thus $\mathcal{O}_P(1)$ is big.

(4) Obvious.

□

Lemma 2. *Suppose the genus $g(C) \geq 2$. Let ω_X be the canonical invertible sheaf over X of general type. Let $\mathcal{A} = \omega_X^{\otimes b} \otimes \mathcal{O}_P(1)$ over $\mathbf{P}(\Omega_X^{\otimes n})$ such that $\mathcal{O}_P(1) \subset \mathcal{A}$ for some $b > 0$. Then there exists the following commutative giagram:*

$$\begin{tikzcd}
\pi^* f^* S^\ell \Omega_C^{\otimes n} \arrow[r] & \pi^* S^\ell \Omega_X^{\otimes n} \arrow[r] & \mathcal{O}_P(\ell) \\
 & \pi^*\pi_*\mathcal{A} \arrow[r] \arrow[u] & \mathcal{A} \arrow[u] \\
\pi^* f^* \Omega_C^{\otimes n} \arrow[r] \arrow[uu] & \pi^* \Omega_X^{\otimes n} \arrow[r] \arrow[u] & \mathcal{O}_P(1) \arrow[u]
\end{tikzcd}$$

Proof. (1) Let $g = \pi \circ f$. $\mathcal{O}_P(1) \otimes g^*\omega_C^{\otimes -n}$ is effective. Hence $\mathcal{O}_P(2) \otimes \omega_C^{\otimes -n}$ is big because $\mathcal{O}_P(1)$ is big.

(2) There exists a number $\ell_0 \geq 1$ such that

$$\mathcal{O}_P \subset \mathcal{A} \otimes g^*\omega_C^{\otimes -n} \subset \otimes^{\ell_0} \left(\mathcal{O}_P(2) \otimes g^*\omega_C^{\otimes -n}\right)$$

(3) Hence

$$g^*\omega_C^{\otimes n} \subset \mathcal{A} \subset \left(\mathcal{O}_P(2\ell_0) \otimes g^*\omega_C^{\otimes -n(\ell_0 - 1)}\right)$$

(4) Thus

$$g^*\omega_C^{\otimes n} \subset \mathcal{A} \subset \mathcal{O}_P(2\ell_0)$$

(5) Set $\ell = 2\ell_0$.

□

It is important to show the commutativity of the diagram above.

We find another approach to this problem without using a Kaehler differential sheaf.

Let k be a field of characteistic 0. Let X and C be non singular projective varieties of general type over k and $f : X \to C$ a projective surjective morphism with connected fibres. Let $\mathcal{E}_b = \omega_X \oplus f^*\omega_C^{\otimes b}$ for $b \geq 1$ and $\pi : \mathbf{P}(\mathcal{E}_b) \to X$ a projective bundle of relative dimension 1 over X.

Lemma 3. *Let X and C be non singular projective varieties of general type over k and $f : X \to C$ a projective surjective morphism with connected fibres. Let $\mathcal{O}_P(1)$ be the fundamental invertible sheaf over $\mathbf{P}(\mathcal{E}_b)$. Then $\mathcal{O}_P(1)$ is big. If ω_X is abundant, $\mathcal{O}_P(1)$ is abundant.*

Proof. Since there exists a natural injection $\omega_X \to \mathcal{E}_b$, we have $\pi^*\omega_X \to \pi^*\mathcal{E}_b \to \mathcal{O}_P(1)$. Applying π_* to the homomorphism above, we get its non triviality. ω_X is big. There is a number b such that $\mathcal{O}_P(1) \otimes \pi^*\omega_X^{\otimes b}$ is big. Hence $\mathcal{O}(b+1) \supset \mathcal{O}_P(1) \otimes \pi^*\omega_X^{\otimes b}$. It implies $\mathcal{O}_P(b+1)$ is big. Thus $\mathcal{O}_P(1)$ is big. If ω_X is abundant, $\mathcal{E}$ is abundant since ω_C is ample. Hence $\mathcal{O}_P(1)$ is abundant. $\square$

When ω_X is abundant and big, $\mathcal{O}_P(1)$ is abundant and big. We have a natural surjective homomorphism for sufficiently large ℓ

$$\mathcal{O}_P \otimes H^0(P, \mathcal{O}_P(\ell)) \to \mathcal{O}_P(\ell).$$

By this we have a morphism

$$\mathbf{P}(\mathcal{E}_b) \to \mathbf{P}(H^0(P, \mathcal{O}_P(\ell))).$$

We denote the image variety of this morphism by Q and the induced morphism by $\rho : P \to Q$. For sufficiently large ℓ, Q is normal and its rational function field $R(Q)$ is algebraically closed in $R(P)$.

Proposition 1. *For a natural projection $\mathcal{E}_b \to f^*\omega_C^{\otimes b}$ there exists a section $\sigma : X \to P$ such that $(\mathcal{E}_b \to f^*\omega_C^{\otimes b}) = (\mathcal{E}_b \to \sigma^*\mathcal{O}_P(1))$. $\sigma(X) \subset P$ is a hypersurface of codimension 1.*

Proof. It is obvious from the universality of the fundamental sheaf $\mathcal{O}_P(1)$.Recall that $\dim P = \dim X + 1$. Hence $\sigma(X)$ is an effective divisor on P. $\square$

Proposition 2. *The morphism $\rho : P \to Q$ maps a divisor $\sigma(X)$ to a curve in Q, which is isomorphic to a curve C.*

Proof. Since $\mathcal{O}_P(1)|_{\sigma(X)} = \pi^*(f^*\omega_C^{\otimes b})|_{\sigma(X)} \cong f^*\omega_C^{\otimes b}$ over $\sigma(X)$, the restriction $\mathcal{O}_P \otimes H^0(P, \mathcal{O}_P(\ell)) \to \mathcal{O}_P(\ell)$ to $\sigma(X)$ is as follows:

$$\begin{array}{ccc}
\mathcal{O}_{\sigma(X)} \otimes H^0(P, \mathcal{O}_P(\ell)) & \longrightarrow & \mathcal{O}_P \otimes H^0(\sigma(X), f^*\omega_C^{\otimes b\ell}) \\
\downarrow & & \downarrow \\
\mathcal{O}_P(\ell)|_{\sigma(X)} & \longrightarrow & f^*\omega_C^{\otimes b\ell}
\end{array}$$

Hence ρ maps $\sigma(X)$ onto a curve in Q which is isomorphic to C. $\square$

Proposition 3. *The hypersurface $\sigma(X) \subset P$ determines an effective divisor which is a holomorphic section of $\mathcal{O}_P(1) \otimes \pi^*\omega_X^{-1}$,i.e. $\mathcal{O}_P(\sigma(X)) = \mathcal{O}_P(1) \otimes \pi^*\omega_X^{-1}$.*

Proof. There is an isomorphism between $P = \mathbf{P}(\omega_X \oplus f^*\omega_C^{\otimes b})$ and $P' = \mathbf{P}(\mathcal{O}_X \oplus \omega_X^{-1} \otimes f^*\omega_C^{\otimes b})$ and a natural isomorphism between fundamental sheaves $\mathcal{O}_{P'}(1) \cong \mathcal{O}_P(1) \otimes \pi^*\omega_X^{-1}$. There is a non void canonical homomorphism

$$\mathcal{O}_P = \pi^*\mathcal{O}_X \to \pi^*(\mathcal{O}_X \oplus f^*\omega_C^{\otimes b} \otimes \omega_X^{-1}) \to \mathcal{O}_{P'}(1) \cong \mathcal{O}_P(1) \otimes \pi^*\omega_X^{-1}$$

Hence an effective divisor determined by the above holomorphic section of $\mathcal{O}_P(1) \otimes \pi^*\omega_X^{-1}$ is a hypersurface $\sigma(X)$. □

Proposition 4. *Let E be the exceptional divisor for a birational morphism $\rho : P \to Q$. Then the intersection number $(E, B_\lambda) \geq 0$, where B_λ is defined by $\mathcal{E}_b|_{C_\lambda} \to \omega_{C_\lambda}^{\otimes b}$ for sufficiently large b.*

Proof. Let G be a hyperplane in $\mathbf{P}(H^0(P, \mathcal{O}_P(\ell)))$. Consider a curve $\rho(X)$ which is isomorphic to C in Q. Remember $\rho(\sigma(X)) = C$. The pull-back ρ^*G is a pull-back of a Cartier divisor for $\rho : P \to Q \subset \mathbf{P}(H^0(P, \mathcal{O}_P(\ell)))$, which is canonically isomorphic to $\mathcal{O}_P(\ell)$. Take a minimal m_0 such that m_0G contains a curve C. Then $\rho^*m_0G = F + E$. Here $\rho_*E = 0$ and $\rho_*F = m_0G$ in the groups of cycle classes $\mathrm{A}_*(Q) = \mathrm{Z}_*(Q)/\mathrm{Rat}_*(Q)$ where $\mathrm{Rat}_*(Q)$ is a group of rationally equivalent to zero cycles on Q. From projection formula we have $(B_\lambda, \rho^*m_0G) = (\rho_*B_\lambda, m_0G) = m_0(C, G)$. We claim $(B_\lambda, F) \leq (C, m_0G)$. The restriction of ρ to B_λ is an isomorphism. Hence every intersection points between B_λ and F projects one to one into C, F may intersects other points with the pull-back of C in Q. Hence $(B_\lambda, F) \leq (C, m_0G)$. Therefore we have $(B_\lambda, E) = (B_\lambda, \rho^*m_0G) - (B_\lambda, F) \geq 0$. □

Proposition 5. *We obtain the following inequality $(\mathcal{O}_P(1) \otimes \pi^*\omega_X^{-1}, B_\lambda) \geq 0$.*

Proof. For any b, $\dim H^0(P, \mathcal{O}_P(\sigma(X)) = \dim H^0(P, \mathcal{O}_P(1) \otimes \pi^*\omega_X^{-1}) = \dim H^0(X, (\omega_X \oplus f^*\omega_C^{\otimes b}) \otimes \omega_X^{-1}) = 1$. Since we have $(E, B_\lambda) \geq 0$ and $B_\lambda \subset \sigma(X) \subset E$, we get $0 \leq (E, B_\lambda) = (E|_{\sigma(X)}, B_\lambda) = (\sigma(X), B_\lambda) = (\mathcal{O}_P(1) \otimes \pi^*\omega_X^{-1}, B_\lambda)$. Remember $\sigma(X)$ is a member of the complete linear system $|\mathcal{O}_P(1) \otimes \pi^*\omega_X^{-1}|$. □

Proposition 6. $(\omega_X, C_\lambda) \leq b(2g(C) - 2)$ *for sufficiently large b.*

Proof. Since $(\mathcal{O}_P(1) \otimes \pi^*\omega_X^{-1}, B_\lambda) \geq 0$, $(\mathcal{O}_P(1), B_\lambda) \geq (\pi^*\omega_X, B_\lambda)$. The fundamental sheaf $\mathcal{O}_P(1)$ is isomorphic to $\omega_{B_\lambda}^{\otimes b} \cong \omega_C^{\otimes b}$ over B_λ. Hence $(\mathcal{O}_P(1), B_\lambda) = b(\omega_{B_\lambda}, B_\lambda) = b(\omega_C, C) = b \deg \omega_C = b(2g(C)-2)$. From projection formula, $(\pi^*\omega_X, B_\lambda) = (\omega_X, \pi_*B_\lambda) = (\omega_X, C_\lambda) \leq b(2g(C) - 2)$. □

Remark 1. *Let $\omega_{P/Q}$ be the relative dualizing sheaf for $\rho : P \to Q$. Since $\omega_{P/Q}|_{\rho^{-1}(C)} \cong \omega_{X/C}$, we have $\omega_{P/Q} = \mathcal{O}_P(-\sigma(X))$. Furthermore, $\pi^*f^*f_*(\omega_{X/C}^{\otimes \ell})|_{B_\lambda} \to \pi^*\omega_{X/C}^{\otimes \ell}|_{B_\lambda}$ is equivalent to $f_*(\omega_{X/C}^{\otimes \ell}) \to \omega_{X/C}^{\otimes \ell}|_{C_\lambda}$. From the weak positivity of $f_*(\omega_{X/C}^{\otimes \ell})$(*[V], [Kaw], [**?**], [Ws]*), $\deg(\omega_{X/C}|_{C_\lambda}) \geq 0$.*

Proposition 7. *Let L be an ample invertible sheaf over X. Then (L, C_λ) is bounded above except for C_λ contained in a fixed hypersurface.*

Proof. There exists a number a such that $L \to \omega_X^{\otimes a}$ is non trivial.Hence $(L, C_\lambda) \leq (\omega_X^{\otimes a}, C_\lambda) \leq ab(2g(C) - 2)$. □

Proposition 8. *There exist a finite number of Hilbert polynomials (*[GG]*) such that for $1 \leq i \leq M$*

$$P_i(m) = \chi(C_\lambda, L^{\otimes m})$$

Hence there exists a Hilbert polynomial $P_i(m)$ such that $P_i(m) = \chi(C_\lambda, L^{\otimes m})$ for curves C_λ which are dense in X.

Proof. It is well known that there correspond a finite number of Hilbert polynomials $\chi(C_\lambda, L^{\otimes m})$ where C_λ are bounded above for an ample invertible sheaf L over X. □

Proposition 9. *There exists a quasi-projective subvariety T of* $\mathrm{Hilb}_X^{P_i(m)}$ *(*[GG]*) such that every poit of T corresponds to a section $C_\lambda \subset X$. We have the following figure:*

$$\begin{array}{ccc} C_\lambda & \subset \Gamma|_T & \subset X \times T \\ \downarrow & \downarrow & \downarrow \\ t & \in T & = T \end{array}$$

Proof. There exists thew following diagram of the universal family Γ over Hilbert scheme $\mathrm{Hilb}_X^{P_i(m)}$.

$$\begin{array}{ccc} C_\lambda & \subset \Gamma & \subset X \times \mathrm{Hilb}_X^{P_i(m)} \\ \downarrow & \downarrow & \downarrow \\ t & \in \mathrm{Hilb}_X^{P_i(m)} & = \mathrm{Hilb}_X^{P_i(m)} \end{array}$$

We can construct a finite number of strata such that each poit of strata corresponds to a section C_λ. There exists a strata T such that $\Gamma \times_{\mathrm{Hilb}_X^{P_i(m)}} T \to X$ is dominant by assumption.

□

Proposition 10. *There exists a dominant rational map*

$$\begin{tikzcd} C \times V \arrow[r] \arrow[d] & X \arrow[dl] \\ C & \end{tikzcd}$$

Proof. There exists an etale cover $T' \to T$ such that $\Gamma_{T'} \to T'$ is a trivial product. Let V be a projective compactification of T'. Then we get the following commutative diagram:

$$\begin{tikzcd}
 & \Gamma|_{T'} \arrow[dl] \arrow[d] \\
C \times V \arrow[r] \arrow[d] & X \arrow[dl] \\
C &
\end{tikzcd}$$

□

Lemma 4. *Let X/C be a fibre space with the generic general fibre of general type. If $V \times C \to X$ over C is dominant, X/C is isotrivial.*

Proof. Since X/C is a fibre space with the generic general fibre of general type, we can apply $\max_{m>0} \kappa(\det f_*\omega_{X/C}^{\otimes m}) \geq \mathrm{var}(X/C)$. We may choose $\dim V = \dim X - 1$ by hyperplane cuts. Thus from the condition that $V \times C \to X$ is dominant, it follows that $\kappa(\det f_*\omega_{X/C}^{\otimes m}) = 0$. Hence $\mathrm{var}(X/C) = 0$, which means that X/C is isotrivial. □

Remark 2. *The abundance conjecture that for a variety of general type there exists a minimal model variety with the canonical sheaf abundant was proved. We can apply it to our case(*[Mats], [Kaw], [Ko], [MP]*).*

3. Arithmetic case

We refer the following definitions to Moriwaki([Mo1], [Mo2], [Szp]).

Definition 1. *A scheme is said to be an arithmetic variety(resp.a projective arithmetic variety)if it is irreducible and reduced scheme and if it is flat and quasi-projective(resp. projective) over*$\mathrm{Spec}\,\mathbf{Z}$. *An arithmetic variety is called generically smooth if the generic fibre $X \times_{\mathrm{Spec}\,\mathbf{Z}} \mathrm{Spec}\,\mathbf{Q}$ is smooth over* $\mathrm{Spec}\,\mathbf{Q}$.

When X is a generically smooth projective arithmetic variety, we have the connected components X_σ for all $\sigma : K \to \mathbf{C}$ of $X(\mathbf{C})$ where we have the Stein factorization $X \to \mathrm{Spec}\,\mathcal{O}_K \to \mathrm{Spec}\,\mathbf{Z}$ for some algebraic field K and the ring of integers of K $\mathcal{O}_K$. We write by $p_x : \mathrm{Spec}\,\mathbf{C} \to X$ the poit $X(\mathbf{C})$ and by $\phi_x : \mathrm{Spec}\,\mathbf{C} \to \mathrm{Spec}\,\mathcal{O}_{X,p_x}$ its homomorphism to the local ring. Let E be a locally free coherent sheaf over X. For each $x \in X(\mathbf{C})$, $E(x) = E_{p_x} \otimes_{\mathcal{O}_{X.p_x}} \mathbf{C}$ is given a hermitian inner product by C^∞-hermitian metric $h = \{h_x\}_{x \in X(\mathbf{C})}$. A couple $\bar{E} = (E, h)$ is called C^∞-hermitian locally free coherent sheaf.

Definition 2. *A C^∞-metric h is said to be of real type if it satisfies the conditionthat for every $x \in X(\mathbf{C})$*

$$h_x(s \otimes^x 1, s' \otimes^x 1) = \overline{h_x(s \otimes^{\bar{x}} 1, s' \otimes^{\bar{x}} 1)}$$

for all $s, s' \in \mathcal{O}_{X,p_x}$. Here $\bar{x}$ is a complex conjugate and $\otimes^x$ means a tensor product with respect to ϕ_x.

Definition 3. *Let X be a generically smooth arithmetic variety, Z a cycle of codimension p and T a $(p-1, p-1)$-current over $X(\mathbf{C})$. A couple (Z, T) is said to be an arithmetic cycle of codimension p. It is called of Green type if $dd^c(T) + \delta_{Z(\mathbf{C})}$ is an element of $A^{p,p}(X(\mathbf{C}))$.*

We refer to the arithmetic projection formula([Mo1]).

Proposition 11. *Let $f : X \to Y$ be a projective morphism between generically smooth arithmetic varieties. $\bar{L} = (L, h)$ a C^∞-hermitian invertible sheaf, s a non zero meromorphic section and η the generic point of X. Suppose that $f(\eta) \notin \mathrm{Supp}(L, s)$ and that $\mathrm{Supp}(f^*L, f^*s)$ and Z intersect properly. Then*

$$f_*(f^*(L), f^*s) \cdot (Z, T)) = (\bar{L}, s) \cdot f_*(Z, T)$$

Let X be an arithmetic variety and F a coherent sheaf. Let $|\cdot|_F$ be a set of norms $\{|\cdot|_{F,x}\}_{x \in X(\mathbf{C})}$

Definition 4. *Let X be an arithmetic variety and $\bar{F} = (F, h)$ C^∞-hermitian coherent sheaf of real type which has a bounded norm.*

(1) *$s \in H^0(X, F)$ is called a small section if $||s||_{sup} \leq 1$,*
(2) *$s \in H^0(X, F)$ is called a strictly small section if $||s||_{sup} < 1$,*

Definition 5. *Let X be an arithmetic variety and $\bar{L} = (L, h)$ a C^∞-hermitian invertible sheaf of real type.*

(1) *$s \in H^0(X, L)$ is called a small section if $||s||_{sup} \leq 1$.*
(2) *$s \in H^0(X, L)$ is called a strictly small section if $||s||_{sup} < 1$.*
(3) *A C^∞-hermitian invertible sheaf $\bar{L} = (L, h)$ is said to be vertically ample if an invertible sheaf L is ample over X and the curvature of L is positive over $X(\mathbf{C})$ with respect to h.*
(4) *A $\bar{L}$ is said to be ample if it is vertically ample and if there exists a number n such that $L^{\otimes n}$ is generated by all strictly small global sections.*
(5) *A $\bar{L}$ is said to be effective it L has a small global section.*
(6) *A $\bar{L}$ is said to be big if there exist an ample C^∞-hermitian invertible sheaf A and a number n such that $\bar{L}^{\otimes n} \otimes \bar{A}^{-1}$ is effective.*
(7) *A $\bar{L}$ is said to be abundant there exist a morphism from X to an arithmetic variety Y such that $\bar{L}^{\otimes n}$ is isomorphic to the pull-back of an ample C^∞-hermitian invertible sheaf over Y for some $n > 0$.*

Proposition 12. *A C^∞-hermitian invertible sheaf $\bar{L} = (L, h)$ is abundant if $L^{\otimes n}$ is generated by its global sections and the curvature of L is semi-positive over $X(C)$ with respect to h and if $L^{\otimes n}$ is generated by its strictly small sections for some $n > 0$.*

Proof. It is enough to show $L^{\otimes n}$ is generated by its strictly small sections. Since $||\cdot||_{Sup,X} \leq || \cdot ||_{Y,Sup}$ for a strictly small global section of $\bar{A}$ over Y, it is obvious. □

Proposition 13. *Let X be a projective smooth arithmetic variety and its Stein factorization $f : X \to \operatorname{Spec} \mathcal{O}_K$. Let $\omega_{X/\mathcal{O}_K}$ be the relative dualizing sheaf and $\bar{\omega}_{X/\mathcal{O}_K} = (\omega_{X/\mathcal{O}_K}, h)$. Suppose $\bar{\omega}_{X/\mathcal{O}_K}$ is abundant, big and $\bar{\omega}_{\mathcal{O}_K}$ is ample. Let $\bar{\mathcal{E}}_b = \bar{\omega}_{X/\mathcal{O}_K} \oplus f^*\bar{\omega}_{\mathcal{O}_K}^{\otimes b}$ and $\pi : \mathbf{P}(\mathcal{E}_b) \to X$ the projective bundle over X. Then there exists a projective morphism $\rho : \mathbf{P}(\mathcal{E}_b) \to \mathbf{P}(\hat{H}^0(X, S^m(\mathcal{E}_b)))$.*

Proof. Let $\mathcal{O}_P(1)$ be the fundamental sheaf and its hermitian structure induced by $\pi^*\mathcal{E}_b \to \mathcal{O}_P(1)$. Then $\mathcal{O}_P \otimes \hat{H}^0(X, S^m\mathcal{E}_b) \to \mathcal{O}_P(m)$ is surjective by assumption. Hence there exists a projective morphism $\rho : \mathbf{P}(\mathcal{E}_b) \to \mathbf{P}(\hat{H}^0(X, S^m(\mathcal{E}_b)))$. □

Proposition 14. *Under the same assumption of the proposition above, we have*

$$\rho_*((\rho^*H, \rho^*s) \cdot (\Delta_x, 0)) = (H.s) \cdot \rho_*(\Delta_x, 0)$$

Here $x \in X(\mathbf{C})$ with $[K(x) : \mathbf{Q}] < \infty$ and s is a meromorphic section of H.

Proof. From the arithmetic projection formula we get it. □

Let Q be an image variety of a morphism $\rho : P \to \mathbf{P}(\hat{H}^0(P, \mathcal{O}_P(m)))$. For sufficiently large m, a variety Q is normal and ρ is a birational morphism. A natural surjective homomorphism $\mathcal{E}_b \to \omega_{\mathcal{O}_K}^{\otimes b}$ determines a section $\sigma : X \to P$. Let $\omega_{P/Q}$ denote the relative dualizing sheaf for $\rho : P \to Q$. We have $\omega_{P/Q}|_{\sigma(X)} \cong \omega_{X/\mathcal{O}_K}$.

Proposition 15. *The restriction $\rho : P \to Q$ to $\sigma(X)$ is a mapping from $\sigma(X)$ onto an arithmetic curve which is isomorphic to* $\operatorname{Spec} \mathcal{O}_K$.

Proof. $\mathcal{O}_P(m)$ on a subvariety $\sigma(X)$ is isomorphic to $\omega_{\mathcal{O}_K}^{\otimes bm}$, which is ample over $\operatorname{Spec} \mathcal{O}_K$. Hence ρ maps $\sigma(X)$ onto an arithmetic curve $\operatorname{Spec} \mathcal{O}_K$ □

Proposition 16. *We have $\mathcal{O}_P(1) = \pi^*\omega_{X/\mathcal{O}_K}(\sigma(X))$ and $\omega_{P/Q} = \mathcal{O}_P(-\sigma(X)) \otimes \pi^* f^* \omega_{\mathcal{O}_K}^{\otimes b}$.*

Proof. There exists an exact sequence $\mathcal{O}_P \to \mathcal{E}_b \otimes \omega_{X/\mathcal{O}_K}^{-1} \to \mathcal{O}_P(1) \otimes \omega_{X/\mathcal{O}_K}^{-1}$. Thus $\sigma(X)$ is a divisor of global section of $\mathcal{O}_P(1) \otimes \omega_{X/\mathcal{O}_K}^{-1}$. We know $\omega_{P/Q} = \mathcal{O}_P(-\sigma(X)) + \pi^* f^* \omega_{\mathcal{O}_K}^{\otimes b}$ since $\omega_{P/Q}|_{\sigma(X)} \cong \omega_{X/\mathcal{O}_K}$ and $\sigma(X)|_{\sigma(X)} = (\mathcal{O}_P(1) - \omega_{X/\mathcal{O}_K})|_{\sigma(X)} = ((1+b)\omega_{\mathcal{O}_K} - \omega_X)|_{\sigma(X)}$ □

Proposition 17. *For a point $x \in X(K)$ and its arithmetic curve $(\Delta_x, 0)$ we have for $\Delta_x \subset \sigma(X)$*

$$\rho_*((\rho^*H, \rho^*s) \cdot (\Delta_x, 0)) = (H.s) \cdot \rho_*(\Delta_x, 0)$$

where $\rho^* H = \mathcal{O}_P(m)$*. Furthermore let* H' *be the strict inverse image of* H*. Then* $\mathcal{O}_P(m) = \mathcal{O}_P(H' + \nu\sigma(X)$*, where* $\nu > 0$ *is a number. We have* $\hat{\deg}\left((\sigma(X), s") \cdot (\Delta_x, 0)\right) \geq 0$*, where* $\rho^* s = s' + s"$*.*

Proof. Since $\hat{\deg}\left((H', s') \cdot (\Delta_x, 0)\right) \leq \hat{\deg}\left((H, s) \cdot (\rho_* \Delta_x, 0)\right)$ for $\Delta_x \subset \sigma(X)$ and $\rho_*((\rho^* H, \rho^* s) \cdot (\Delta_x, 0)) = (H.s) \cdot \rho_*(\Delta_x, 0)$ by projection formula, we get $\hat{\deg}\left((\sigma(X), \rho^* s) \cdot (\Delta_x, 0)\right) \geq 0$ from $(\rho^* H, \rho^* s) \cdot (\Delta_x, 0) = (H', s') \cdot (\Delta_x, 0) + (\sigma(X), s") \cdot (\Delta_x, 0)$ □

Proposition 18. $\hat{\deg}(\bar{\omega}_X) \leq (1 + b)\hat{\deg}(\bar{\omega}_{\mathcal{O}_K})$ *for sufficiently large* b*.*

Proof. It follows from $\hat{\deg}(b\bar{\omega}_{\mathcal{O}_K} - \bar{\omega}_{X/\mathcal{O}_K}) \geq 0$. □

Proposition 19. *Let* $\bar{L}$ *be a* C^∞ *hermitian ample invertible sheaf over* X*, There exists a number* ℓ *such that* $\bar{\omega}_X^{\otimes \ell} \geq \bar{L}$*.*

Proof. Since $\bar{\omega}_X$ is big, it is obvious. □

Definition 6.

$$h_{X,\bar{L}} = \frac{\hat{\deg}(\bar{L}|_{\Delta_x})}{[K(x), \mathbf{Q}]}$$

We apply the following lemma to the proposition above to get the conjecture.

Lemma 5 (Northcott([Szp]))**.** *Let* X *be a projective arithmetic variety over* Spec *mathcalO*$_K$ *and* $\bar{L}$ *an ample invertible sheaf over* X*. Let* $h_{\bar{L}} : X(\bar{K}) \to \mathbf{R}$ *denote the height function associated to* $\bar{L}$*. Then for any positive real numbers* ϵ*,* M*, the set* $\{x \in X(\bar{K}) | [K(x) : \mathbf{Q}] \leq \epsilon, h_{\bar{L}}(x) \leq M\}$ *is a finite set.*

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