

Postcritical sets and saddle basic sets for Axiom A polynomial skew products

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In this note, the dynamics of Axiom A polynomial skew products on \mathbb{C}^2 is investigated. Especially, the postcritical sets are described in terms of the saddle basic sets.

1 Introduction

In this note, we consider Axiom A regular polynomial skew products on \mathbb{C}^2 . It is of the form : $f(z, w) = (p(z), q(z, w))$, where $p(z) = z^d + \cdots$ and $q_z(w) = q(z, w) = w^d + \cdots$ are polynomials of degree $d \geq 2$. Then its k -th iterate is expressed by :

$$f^k(z, w) = (p^k(z), q_{p^{k-1}(z)} \circ \cdots \circ q_z(w)) =: (p^k(z), Q_z^k(w)).$$

Hence it preserves the family of fibers $\{z\} \times \mathbb{C}$ and this makes it possible to study its dynamics more precisely. Let K be the set of points with bounded orbits and put $K_z := \{w \in \mathbb{C}; (z, w) \in K\}$ and $K_{J_p} := K \cap (J_p \times \mathbb{C})$. The *fiber Julia set* J_z is the boundary of K_z .

Let Ω be the set of *non-wandering points* for f . Then f is said to be *Axiom A* if Ω is compact, hyperbolic and periodic points are dense in Ω . For polynomial skew products, Jonsson [J2] has shown that f is Axiom A if and only if the following three conditions are satisfied :

- (a) p is hyperbolic,
- (b) f is vertically expanding over J_p ,
- (c) f is vertically expanding over $A_p := \{\text{attracting periodic points of } p\}$.

Here f is *vertically expanding over* $Z \subset \mathbb{C}$ with $p(Z) \subset Z$ if there exist $\lambda > 1$ and $C > 0$ such that $|(Q_z^k)'(w)| \geq C\lambda^k$ holds for any $z \in Z, w \in J_z$ and $k \geq 0$.

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We are interested in the dynamics of f on $J_p \times \mathbb{C}$ because the dynamics outside $J_p \times \mathbb{C}$ is fairly simple. Consider the critical set

$$C_{J_p} = \{(z, w) \in J_p \times \mathbb{C}; q'_z(w) = 0\}$$

over the *base Julia set* J_p . Let μ be the ergodic measure of maximal entropy for f (see Fornaess and Sibony [FS1]). Its support J_2 is called the *second Julia set* of f . Let $PC_{J_p} := \bigcup_{n \geq 1} f^n(C_{J_p})$ be the *postcritical set* of C_{J_p} . Jonsson [J2] has shown that

- (d) $J_2 = \overline{\bigcup_{z \in J_p} \{z\} \times J_z}$ (Corollary 4.4),
- (e) the condition (b) $\iff PC_{J_p} \cap J_2 = \emptyset$ (Theorem 3.1),
- (f) J_2 is the closure of the set of repelling periodic points of f (Corollary 4.7).

By the Birkhoff ergodic theorem, μ -a.e. x has a dense orbit in J_2 . Especially, $J_2 = \text{supp } \mu$ is transitive. Hence J_2 coincides with the *basic set* of unstable dimension two. See also [FS2].

For any subset X in \mathbb{C}^2 , its accumulation set is defined by

$$A(X) = \bigcap_{N \geq 0} \overline{\bigcup_{n \geq N} f^n(X)}.$$

DeMarco & Hruska [DH1] defined the *pointwise* accumulation set by

$$A_{pt}(C_{J_p}) = \overline{\bigcup_{x \in C_{J_p}} A(x)}.$$

Let Λ be the closure of the set of saddle periodic points in $J_p \times \mathbb{C}$. It decomposes into a disjoint union of *saddle basic sets* : $\Lambda = \sqcup_{i=1}^m \Lambda_i$. Put $\Lambda_z = \{w \in \mathbb{C}; (z, w) \in \Lambda\}$. The *stable* and *unstable manifolds* of Λ are respectively defined by

$$W^s(\Lambda) = \{y \in \mathbb{C}^2; f^k(y) \rightarrow \Lambda\},$$

$$W^u(\Lambda) = \{y \in \mathbb{C}^2; \exists \text{ backward orbit } \hat{y} = (y_{-k}) \text{ tending to } \Lambda\}.$$

Theorem A ([DH1]).

$$A_{pt}(C_{J_p}) = \Lambda, \quad A(C_{J_p}) = W^u(\Lambda) \cap (J_p \times \mathbb{C}).$$

We define a relation \succ among saddle basic sets by $\Lambda_i \succ \Lambda_j$ if $(W^u(\Lambda_i) \setminus \Lambda_i) \cap (W^s(\Lambda_j) \setminus \Lambda_j) \neq \emptyset$. A *cycle* is a chain of basic sets : $\Lambda_{i_1} \succ \Lambda_{i_2} \succ \cdots \succ \Lambda_{i_n} = \Lambda_{i_1}$. For Axiom A open endomorphisms, there is no trivial cycle because $W^u(\Lambda_i) \cap W^s(\Lambda_i) = \Lambda_i$ holds for any i . See [J2], Proposition A.4. Jonsson has also shown that, for Axiom A polynomial skew products on \mathbb{C}^2 , the non-wandering set Ω coincides with the *chain recurrent set* \mathcal{R} . This leads to the following lemma, which we use later.

Lemma 1.1. ([J2], Corollary 8.14) *Axiom A polynomial skew products on \mathbb{C}^2 have no cycles.*

Put

$$C_0 := C_{J_p} \setminus K_{J_p}, \quad C_i := C_{J_p} \cap W^s(\Lambda_i) \quad (1 \leq i \leq m).$$

Lemma 1.2. $C_{J_p} = \sqcup_{i=0}^m C_i$.

proof. By Proposition 3.3 in Jonsson [J1], $\hat{\Omega}$ has local product structure for open Axiom A endomorphisms. Theorem A implies $A(x) \subset \Lambda$ for any $x \in C_{J_p}$. If $A(x) = \emptyset$, then $x \in C_0$. Otherwise there exist an n and $y \in \Lambda$ such that $f^n(x) \in W_{loc}^s(y)$. Hence $A(x) \subset \Lambda_i$ if $y \in \Lambda_i$. Thus we conclude $C_{J_p} = \sqcup_{i=0}^m C_i$. \square

We put $\Lambda_0 = \{[0 : 1 : 0]\}$, which is a superattracting fixed point of f in \mathbb{P}^2 . Hence it is not in the saddle set Λ . We also define $W^s(\Lambda_0) = \{x \in J_p \times \mathbb{C}; f^n(x) \rightarrow \Lambda_0\}$. For $x \in (J_p \times \mathbb{C}) \setminus K_{J_p}$, $f^n(x)$ tends to $[0 : 1 : 0]$. Then we have $A(C_i) \supset A_{pt}(C_i) = \Lambda_i$ for any $i \geq 0$.

Theorem B (Nakane [N]). *For any $i \geq 0$, we have*

$$A(C_i) = \Lambda_i \iff C_i \text{ is closed.}$$

The main purpose of this note is to consider the case where C_i is not necessarily closed. Put $M = \{0, 1, 2, \dots, m\}$ and $I_i = \{j \in M; \overline{C_i} \cap C_j \neq \emptyset\}$.

Theorem 1.1. $A(C_i) \subset W^u(\cup_{j \in I_i} \Lambda_j) \cap W^s(\cup_{j \in I_i} \Lambda_j) \quad (0 \leq i \leq m)$.

If C_i is closed, then $I_i = \{i\}$ and Theorem 1.1 says

$$A(C_i) \subset W^u(\Lambda_i) \cap W^s(\Lambda_i) = \Lambda_i.$$

Since $A(C_i) \supset \Lambda_i$, Theorem 1.1 generalizes Theorem B.

2 Proof of Theorem

First we will give preliminary lemmas. Take a small open neighborhood U_k of Λ_k for $1 \leq k \leq m$ so that $f(U_k) \cap U_\ell = \emptyset$ for $k \neq \ell$. The argument in the proof of the following lemma will be frequently used later.

Lemma 2.1. *Suppose $i \neq j$ and a sequence $\{x_n\}$ in C_i converges to a point x_0 in C_j . Then $A(\{x_n; n \geq 1\})$ contains a point y in $W^u(\Lambda_j) \setminus \Lambda$.*

proof. Since $x_0 \in C_j$, there exists a $K > 1$ such that $f^k(x_0) \in U_j$ for $k \geq K$. Then $f^K(x_n) \in U_j$ for large n . Since $x_n \in C_i$, the orbit of x_n eventually leaves U_j . Hence put $k_n := \min\{k \geq K; f^k(x_n) \notin U_j\}$. We will show $k_n \rightarrow \infty$. Otherwise, taking a subsequence, we may assume $\{k_n\}$ is bounded. Then there is a subsequence n_ℓ such that $k_{n_\ell} = L$ for all ℓ . That is, $f^L(x_{n_\ell}) \notin U_j$. Taking $\ell \rightarrow \infty$, we have $f^L(x_0) \notin U_j$, which contradicts $L \geq K$. Now let y be an accumulation point of the sequence $\{f^{k_n}(x_n)\}$. From the definition of U_k , we have $y \notin \cup U_k$, hence $y \notin \Lambda$. Thus the set $A(\{x_n; n \geq 1\})$ contains a point y outside Λ .

Next we show $y \in W^u(\Lambda_j)$. In fact, by taking subsequences if necessary, put $y_{-\ell} = \lim_{n \rightarrow \infty} f^{k_n - \ell}(x_n)$. Then $\{y_{-\ell}; \ell \geq 0\}$ forms a backward orbit of y in $\overline{U_j}$. By the local product structure of $\hat{\Omega}$, we conclude $y_{-\ell} \rightarrow \Lambda_j$, hence $y \in W^u(\Lambda_j)$. This completes the proof. \square

Lemma 2.2. *If C is a closed set contained in C_i , then $A(C) \subset \Lambda_i$.*

proof. If $C \subset C_0$, then $A(C) = \emptyset$ since C is compact. Suppose $C \subset C_i$ and there exists $x \in A(C) \setminus \Lambda_i$ for $i \geq 1$. By taking U_i small, there exists a neighborhood V of x such that $V \cap U_i = \emptyset$. Since $x \in \cup_{k \geq N} f^k(C)$ for any $N \geq 0$, there exist $m_n \nearrow \infty$ and $x_n \in C$ such that $f^{m_n}(x_n) \in V$, i.e. $f^{m_n}(x_n) \notin U_i$ for any n . Since C is closed, we may assume x_n tends to some $x_0 \in C \subset C_i$. By the same argument as above, if we put $k_n := \min\{k \geq K; f^k(x_n) \notin U_i\}$, we have an accumulation point y of $\{f^{k_n}(x_n)\}$ in $W^u(\Lambda_i) \setminus \Lambda$. We have $y \notin W^s(\Lambda_i)$ because $W^u(\Lambda_i) \cap W^s(\Lambda_i) = \Lambda_i$. Since $y \in A(C)$, $y \in K_{J_p} \setminus J_2 = W^s(\Lambda)$. Thus y must belong to $W^s(\Lambda_{i_1})$ for some $i_1 \neq i$. That is, we have a sequence $\{f^{k_n}(x_n)\}$ in $W^s(\Lambda_i)$ tending to $y \in (W^u(\Lambda_i) \setminus \Lambda_i) \cap (W^s(\Lambda_{i_1}) \setminus \Lambda_{i_1})$, hence we have an order $\Lambda_i \succ \Lambda_{i_1}$.

By successively applying this argument, we have a chain of saddle basic sets :

$$\Lambda_i \succ \Lambda_{i_1} \succ \Lambda_{i_2} \succ \cdots, \quad i \neq i_1 \neq i_2 \neq \cdots.$$

Since there exist only finitely many basic sets, we must have a cycle of them, which contradicts Lemma 1.1. \square

Proposition 2.1. *For any $i, j \in M$, the following four conditions are equivalent to each other.*

- (1) $\overline{C_i} \cap C_j \neq \emptyset$,
- (2) $A(C_i) \cap W^s(\Lambda_j) \neq \emptyset$,
- (3) $\overline{W^s(\Lambda_i)} \cap W^s(\Lambda_j) \neq \emptyset$,
- (4) $W^s(\Lambda_i) \cap W^u(\Lambda_j) \neq \emptyset$.

proof. Since, for any $i \geq 1$, all the sets $\overline{C_i} \cap C_0$, $A(C_i) \cap W^s(\Lambda_0)$, $\overline{W^s(\Lambda_i)} \cap W^s(\Lambda_0)$ and $W^s(\Lambda_i) \cap W^u(\Lambda_0)$ are empty, we may assume $j \geq 1$.

((1) \Rightarrow (2)) Suppose $\overline{C_i} \cap C_j \neq \emptyset$. We will show $A(C_i) \cap \Lambda_j \neq \emptyset$. Let $x_n \in C_i$ tending to $p \in C_j$. Then there exist $y \in \Lambda_j$ and an k such that $f^k(p) \in W_{loc}^s(y)$. Hence, for any $n > 0$, there exists an L_n such that $d(f^\ell(p), f^{\ell-k}(y)) < 1/n$ for $\ell \geq L_n$. For each fixed n , take k_n so that $d(f^{L_n}(x_{k_n}), f^{L_n}(p)) < 1/n$. Then it follows $d(f^{L_n}(x_{k_n}), f^{L_n-k}(y)) < 2/n$. Since $f^{L_n-k}(y) \in \Lambda_j$, we conclude that $A(C_i)$ intersects Λ_j .

((2) \Rightarrow (3)) It is evident since $A(C_i) \subset \overline{W^s(\Lambda_i)}$.

((3) \Rightarrow (1)) Suppose $W^s(\Lambda_i) \ni x_n \rightarrow p \in W^s(\Lambda_j)$. Then there exist k and $y = (z_0, w_0) \in \Lambda_j$ such that $f^k(p) \in W_{loc}^s(y)$. Let U_y be the connected component of $\text{int } K_{z_0}$ containing w_0 . Then $U_y \supset W_{loc}^s(y)$ and there exists $L > 0$ such that $f^L(U_y) = U_{f^L(y)}$ contains a critical point c in C_j . If we put $(z_n, w_n) = f^k(x_n)$, then $z_n \in J_p$. By the continuity of the map $z \mapsto J_z$ in J_p , $f^L(U_{f^k(x_n)}) \rightarrow f^L(U_{f^k(p)}) = f^L(U_y)$. Thus, for large n , there exist critical points $c_n \in f^L(U_{f^k(x_n)})$ tending to c . Since $c_n \in C_i$, we conclude that $\overline{C_i} \cap C_j \neq \emptyset$.

((3) \Rightarrow (4)) Suppose $W^s(\Lambda_i) \ni x_n \rightarrow p \in W^s(\Lambda_j)$. By the above argument, we have $U_{f^k(x_n)} \rightarrow U_{f^k(p)} = U_y$. Since, for any prehistory $\hat{y} \in \hat{\Lambda}_j$ of y , $W_{loc}^u(\hat{y})$ is transversal to the fiber, it follows that $U_{f^k(x_n)} \cap W_{loc}^u(\hat{y}) \neq \emptyset$ for large n . Thus we conclude $W^s(\Lambda_i) \cap W^u(\Lambda_j) \neq \emptyset$.

((4) \Rightarrow (3)) We need a lemma.

Lemma 2.3. *Suppose $\overline{W^s(\Lambda_i)} \cap W^s(\Lambda_j) = \emptyset$. Then there exists $\delta > 0$ such that $W^s(\Lambda_i) \cap W_\delta^u(\hat{y}) = \emptyset$ for any $\hat{y} \in \hat{\Lambda}_j$.*

proof. Suppose, for any $n \geq 1$, there exist $\hat{y}_n \in \hat{\Lambda}_j$ and $x_n \in W^s(\Lambda_i) \cap W_{1/n}^u(\hat{y}_n)$. Since $\{y_n\} \subset \Lambda_j$, it follows that $\{x_n\}$ and $\{y_n\}$ are bounded. Hence there exist their subsequences $\{x_{n_k}\}$ and $\{y_{n_k}\}$ respectively tending to x_0 and y_0 . Since $d(x_n, y_n) < 1/n$, we have $x_0 = y_0 \in \Lambda_j$. Thus we conclude $\overline{W^s(\Lambda_i)} \cap W^s(\Lambda_j) \neq \emptyset$. \square

Now suppose $W^s(\Lambda_i) \cap W^u(\Lambda_j) \neq \emptyset$. If we take $p \in W^s(\Lambda_i) \cap W^u(\Lambda_j)$, there exists a prehistory $\hat{p} = (p_{-n})$ of p tending to Λ_j . Assume $\overline{W^s(\Lambda_i)} \cap W^s(\Lambda_j) = \emptyset$. Let $\delta > 0$ be the constant in Lemma 2.3. Then there exists $L > 0$ such that $d(p_{-n}, \Lambda_j) < \delta$ for any $n \geq L$, hence, $p_{-L} \in W_\delta^u(\hat{y})$ for some $\hat{y} \in \hat{\Lambda}_j$. Since $p_{-L} \in W^s(\Lambda_i)$, we have $W^s(\Lambda_i) \cap W_\delta^u(\hat{y}) \neq \emptyset$, which contradicts Lemma 2.3. This completes the proof of Proposition 2.1. \square

Corollary 2.1. $A(C_i) \subset \cup_{j \in I_i} W^s(\Lambda_j) \quad (0 \leq i \leq m).$

proof. By Lemma 3.5 in [DH2], we have $W^u(\Lambda) \cap K_{J_p} = W^u(\Lambda) \cap W^s(\Lambda)$. Then, for $i \geq 1$ it follows $A(C_i) \subset W^u(\Lambda) \cap K_{J_p} \subset W^s(\Lambda)$. Now the assertion follows from Proposition 2.1. \square

Now we prove Theorem 1.1. We give a proof mainly for the case $i \geq 1$. The case $i = 0$ can be done with a minor change and will be given at the end.

Suppose $p \in A(C_i)$. Then there exists $x_n \in C_i$ and $m_n \nearrow \infty$ such that $f^{m_n}(x_n) \rightarrow p$. If $p \in \Lambda_j$ for some j , then $A(C_i) \cap W^s(\Lambda_j) \neq \emptyset$. Hence we have $j \in I_i$ by Proposition 2.1. Thus, if $p \in \Lambda$, then $p \in \cup_{j \in I_i} \Lambda_j$. This holds also for $i = 0$.

In the sequel, we assume $p \notin \Lambda$. We may assume $p \notin \overline{\cup_{j=1}^m U_j}$. We also assume $x_n \rightarrow x_0$. Suppose $x_0 \in C_i$. Then by Lemma 2.2, we have $p \in \Lambda_i$, which contradicts $p \notin \Lambda$. Hence $x_0 \in C_{i_1}$ for some $i_1 \neq i$ and it follows $i_1 \in I_i$. As in the proof of Lemma 2.1, we take K_1 so that $f^k(x_0) \in U_{i_1}$ for $k \geq K_1$ and put $k_n^{(1)} = \min\{k \geq K_1; f^k(x_n) \notin U_{i_1}\}$. If $m_n < k_n^{(1)}$ for infinitely many n , then $p = \lim f^{m_n}(x_n) \in \overline{U_{i_1}}$, a contradiction. Thus $m_n \geq k_n^{(1)}$ for large n . We may assume $f^{k_n^{(1)}}(x_n)$ tends to some $y^{(1)} \in W^u(\Lambda_{i_1}) \setminus \Lambda$. Suppose $y^{(1)} \in W^s(\Lambda_{i_2})$. Since $y^{(1)} \in A(C_i)$, we have $i_2 \in I_i$.

We repeat the argument in the proof of Lemma 2.1. Put K_2 so that $f^k(y^{(1)}) \in U_{i_2}$ for $k \geq K_2$ and $k_n^{(2)} = \min\{k \geq k_n^{(1)} + K_2; f^k(x_n) \notin U_{i_2}\}$. If $k_n^{(1)} \leq m_n \leq k_n^{(1)} + K_2$ for infinitely many n , there exists $j \leq K_2$ so that $m_n = k_n^{(1)} + j$ for infinitely many n . Then we have

$$p = \lim f^{m_n}(x_n) = \lim f^{k_n^{(1)}+j}(x_n) = f^j(y^{(1)}) \in W^u(\Lambda_{i_1}).$$

Otherwise, we have $m_n \geq k_n^{(1)} + K_2$ for large n . We may assume $f^{k_n^{(2)}}(x_n) \rightarrow y^{(2)} \in W^u(\Lambda_{i_2}) \setminus \Lambda$. If $m_n < k_n^{(2)}$ for infinitely many n , then $p = \lim f^{m_n}(x_n) \in \overline{U_{i_2}}$, a contradiction. Thus $m_n \geq k_n^{(2)}$ for large n .

Repeating this argument, we eventually meet Λ_i . That is, there exist ℓ and $i_j \in I_i, 1 \leq j \leq \ell$ such that

$$\Lambda_{i_1} \succ \Lambda_{i_2} \succ \cdots \succ \Lambda_{i_\ell} = \Lambda_i.$$

Suppose $k_n^{(\ell)} < \infty$ for infinitely many n . Then, further repeating this argument, we must meet Λ_i again. That is, there exists a sequence :

$$\Lambda_{i_1} \succ \cdots \succ \Lambda_{i_\ell} = \Lambda_i \succ \cdots \succ \Lambda_i.$$

This contradicts Lemma 1.1. Thus we conclude $k_n^{(\ell)} = \infty$ and $f^k(x_n) \in U_{i_\ell}$ ($k \geq k_n^{(\ell-1)} + K_\ell$) for large n . Since $p \notin \overline{U_{i_\ell}}$, we may conclude $k_n^{(\ell-1)} \leq m_n \leq k_n^{(\ell-1)} + K_\ell$ for large n . Then, there exists $j \leq K_\ell$ such that $m_n = k_n^{(\ell-1)} + j$ for infinitely many n and

$$p = \lim f^{m_n}(x_n) = \lim f^{k_n^{(\ell-1)}+j} = f^j(y^{(\ell-1)}) \in W^u(\Lambda_{i_{\ell-1}}).$$

Thus we conclude $A(C_i) \subset \cup_{j \in I_i} (W^u(\Lambda_j) \cap (J_p \times \mathbb{C}))$. Combining this with Corollary 2.1, we have

$$A(C_i) \subset W^u(\cup_{j \in I_i} \Lambda_j) \cap W^s(\cup_{j \in I_i} \Lambda_j).$$

Let us consider the case $i = 0$. Let $p \in A(C_0)$. The argument as above works as long as $y^{(\ell)} \in W^s(\Lambda)$. Suppose $y^{(\ell)} \notin W^s(\Lambda)$. Then it belongs to $W^s(\Lambda_0)$. If $m_n \leq k_n^{(\ell)}$, then $p \in \overline{U_{i_\ell}}$, a contradiction. Now suppose $0 < m_n - k_n^{(\ell)} < K_{\ell+1}$ for some $K_{\ell+1} > 1$. Then there exists $j \geq 1$ such that $m_n = k_n^{(\ell)} + j$ for infinitely many n and $p = \lim f^{m_n}(x_n) = f^j(y^{(\ell)}) \in W^u(\Lambda_{i_\ell})$. Next suppose the sequence $m_n - k_n^{(\ell)}$ is unbounded. By taking a subsequence tending to ∞ , we have

$$p = \lim f^{m_n - k_n^{(\ell)}} \circ f^{k_n^{(\ell)}}(x_n) = [0 : 1 : 0].$$

which implies $p \in W^u(\Lambda_0) = \{[0 : 1 : 0]\}$. This completes the proof of Theorem 1.1. \square

Note that C_j is open in C_{J_p} if and only if $\overline{C_i} \cap C_j = \emptyset$ for any $i \neq j$. After Proposition 2.1, it is equivalent to $\overline{W^s(\Lambda_i)} \cap W^s(\Lambda_j) = \emptyset$ for any $i \neq j$.

Theorem 2.1. *For any fixed $j \geq 1$, we have*

$$C_j \text{ is open in } C_{J_p} \iff \text{the map } z \mapsto \Lambda_{j,z} \text{ is continuous in } J_p,$$

which is also equivalent to $W^u(\Lambda_j) \cap (J_p \times \mathbb{C}) = \Lambda_j$.

proof. (\Rightarrow) Suppose the map $z \mapsto \Lambda_{j,z}$ is discontinuous at $z_0 \in J_p$. Then there exist $y = (z_0, w_0) \in \Lambda_j$, a sequence $\{z_n\}$ in J_p tending to z_0 and $\epsilon > 0$ such that $\mathbb{D}(w_0, \epsilon) \cap \Lambda_{j,z_n} = \emptyset$ for any $n \geq 1$. By Lemma 2.3, there exists $\delta > 0$ such that, for any prehistory $\hat{y} \in \hat{\Lambda}_j$ of y ,

$$\begin{aligned} W_\delta^u(\hat{y}) \cap (J_p \times \mathbb{C}) \setminus K_{J_p} &= W_\delta^u(\hat{y}) \cap (J_p \times \mathbb{C}) \cap W^s(\Lambda_0) = \emptyset, \\ W_\delta^u(\hat{y}) \cap (J_p \times \mathbb{C}) \cap K_{J_p} &= W_\delta^u(\hat{y}) \cap W^s(\Lambda) = W_\delta^u(\hat{y}) \cap W^s(\Lambda_j) \subset \Lambda_j. \end{aligned}$$

Hence we have $W_\delta^u(\hat{y}) \cap (J_p \times \mathbb{C}) \subset \Lambda_j$. This contradicts the fact $\mathbb{D}(w_0, \epsilon) \cap \Lambda_{j, z_n} = \emptyset$ for large n since $W_\delta^u(\hat{y})$ is transversal to the vertical fiber.

(\Leftarrow) Suppose C_j is not open in C_{J_p} . Then there exist $i \neq j$ and a sequence x_n in $W^s(\Lambda_i)$ tending to $p \in W^s(\Lambda_j)$. Take $k \geq 1$ and $y = (z_0, w_0) \in \Lambda_j$ so that $f^k(p) \in W_{loc}^s(y)$. Note that $(z_n, w_n) := f^k(x_n) \in W^s(\Lambda_i)$ tends to $f^k(p)$. By the continuity of the map $z \mapsto J_z$, we conclude that $\mathbb{D}(w_0, \epsilon) \cap \Lambda_{j, z_n} = \emptyset$ for large n and for small $\epsilon > 0$. This contradicts the continuity of the map $z \mapsto \Lambda_{j, z}$. \square

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