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Dynamics of a family of regular polynomial maps of $\mathbb{C}^2$

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In this note, the dynamics of regular polynomial endmorphisms of $\mathbb{C}^2$ is investigated. Especially, their Böttcher coordinates are constructed.

1 Introduction

In this note, we will construct the Böttcher coordinates for maps F_c of the form :

$$F_c(x,y) = (x^2 - cy, y^2 - cx),$$

which is studied in Uchimura [U].

Let $f(z)$ be a polynomial endomorphism of $\mathbb{C}^k$ of degree d and let $f_h(z)$ be the degree d part of $f(z)$. It is *regular* if $f_h^{-1}(0) = \{0\}$. Note that regular polynomial maps extend to analytic maps of $\mathbb{P}^k$. Let Π denote the hyperplane at ∞, which is isomorphic to $\mathbb{P}^{k-1}$. In case $k = 2$, Π is isomorphic to the Riemann sphere $\overline{\mathbb{C}}$. For a regular polynomial map f, we denote the *filled-in Julia set* also by $K(f)$. It is a compact subset of $\mathbb{C}^k$. And $J(f)$ denotes the *smallest Julia set* of f, that is, the support of $\mu = (dd^cG_f)^k$. Here G_f is the *Green function* of f :

$$G_f(z) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ |f^n(z)|,$$

which expresses the escape rate of the orbit of the point $z \in \mathbb{C}^k$. We put $f_\Pi = f|_\Pi$, $J_\Pi = J(f_\Pi)$ and let $\mathcal{C}(f)$ be the set of the critical points of f. We define the *stable sets* $W^s(J_\Pi, f)$, $W^s_{loc}(\zeta)$ and $W^s(\zeta)$ of J_Π and $\zeta \in J_\Pi$ respectively by

$$\begin{aligned} W^s(J_\Pi, f) &= \{z \in \mathbb{P}^k; \lim_{n\to\infty} d(f^n(z), J_\Pi) = 0\}, \\ W^s_{loc}(\zeta, f) &= \{z \in \mathbb{P}^k; d(f^n(z), f^n(\zeta)) < \delta,\ n \geq 0\}, \\ W^s(\zeta, f) &= \{z \in \mathbb{P}^k; \lim_{n\to\infty} d(f^n(z), f^n(\zeta)) = 0\}. \end{aligned}$$

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For example, the map F_c above is regular, $F_{c,h}(x,y) = (x^2, y^2) = F_0(x,y)$ and

$$\begin{aligned} K(F_0) &= \{|x| \le 1, |y| \le 1\}, \quad J(F_0) = \{|x| = |y| = 1\}, \\ \mathcal{C}(F_c) &= \{xy = 4c^2\}, \quad F_{c,\Pi}(\zeta) = \zeta^2, \quad J(F_{c,\Pi}) = \{|\zeta| = 1\} \quad \text{(for any } c), \\ W^s(\zeta, F_0) &= \{y = \zeta x, \ |x| > 1\}, \quad W^s(J_\Pi, F_0) = \{|x| = |y| > 1\}. \end{aligned}$$

Note that $F_{c,\Pi}$ is uniformly expanding on J_Π.

Put $A_0 = \{z \in \mathbb{P}^k; G_f(z) > R_0\}$ and $W_0^s(a) = W_{loc}^s(a) \cap A_0$, $W_0^s(J_\Pi, f) = W^s(J_\Pi, f) \cap A_0$. Note that $W_0^s(a)$ is a complex disk. The *Böttcher coordinate* Φ of f was defined in [BJ] as a homeomorphism $W_0^s(J_\Pi, f) \to W_0^s(J_\Pi, f_h)$ satisfying $\Phi \circ f = f_h \circ \Phi$. It extends to $W^s(J_\Pi, f)$ until it meets a critical point.

Bedford and Jonsson [BJ] have constructed the Böttcher coordinates for general regular polynomial endomorphisms on $\mathbb{C}^k$. See also Hubbard and Papadopol [HP] and Peng [P]. Here we will give a more direct and elementary construction for our maps F_c. Then we can construct their Böttcher coordinates as holomorphic maps in an open neighborhood of $W_0^s(J_\Pi, F_c)$ and we can show the uniqueness in some sense.

2 Böttcher coordinates for maps F_c

Put $(x_n, y_n) = F_c^n(x,y)$ and consider the orbits of points in the region

$$D_{k,R} = \left\{ (x,y) \in \mathbb{C}^2; \frac{1}{k}|x| \le |y| \le k|x|, \quad |x|, \ |y| \ge 2R \right\}.$$

Note that, for large k and R, each connected component of the set $\{(x,y) \in \mathbb{C}^2; |x|, |y| \ge 2R\} \setminus D_{k,R}$ is contained in the basin of the super-attracting fixed point $[1:0:0]$ or $[0:1:0]$ in Π. Thus it follows $W_0^s(J_\Pi, F_c) \subset D_{k,R}$. In the sequel, we always assume $k > 1$ and

$$R \ge 3k \cdot \left(\max_{n \ge 0} \left(\frac{|c|}{3} \right)^{1/2^n} + 1 \right). \tag{2.1}$$

Lemma 2.1. *For $(x,y) \in D_{k,R}$ and for $n \ge 1$, it follows*

$$\begin{aligned} |x_n|, \ |y_n| &\ge 2R^{2^n}, \quad |x_n| \ge \frac{|x_{n-1}|^2}{2}, \quad |y_n| \ge \frac{|y_{n-1}|^2}{2}, \\ |y_n| &\le \frac{1}{3}(3k)^{2^n}|x_n|, \quad |x_n| \le \frac{1}{3}(3k)^{2^n}|y_n|. \end{aligned}$$

proof. We prove the lemma by induction on n. The case $n = 1$ follows from the following.

$$\begin{aligned}
|x_1| &\geq |x|^2 - |cy| \geq |x|^2 \left(1 - \frac{|c|k}{|x|}\right) \left(\geq \frac{|x|^2}{2}\right) \\
&\geq 4R^2 \left(1 - \frac{|c|k}{2R}\right) \geq 2R^2, \\
|y_1| &\leq |y|^2 + |cx| \leq |y|^2 \left(1 + \frac{|c|k}{|y|}\right) \\
&\leq |y|^2 \left(1 + \frac{|c|k}{2R}\right) \leq \frac{3k^2|x|^2}{2} \leq 3k^2|x_1|.
\end{aligned}$$

Now suppose the case n and we prove the case $n+1$.

$$\begin{aligned}
|x_{n+1}| &\geq |x_n|^2 \left(1 - \frac{|cy_n|}{|x_n|^2}\right) \geq |x_n|^2 \left(1 - \frac{(3k)^{2^n}|c|}{3|x_n|}\right) \\
&\geq |x_n|^2 \left(1 - \frac{(3k)^{2^n}|c|}{6R^{2^n}}\right) \left(\geq \frac{|x_n|^2}{2}\right) \\
&\geq 2R^{2^{n+1}}.
\end{aligned}$$

Here we use the assumption (2.1). By the same way, we have

$$|y_{n+1}| \leq |y_n|^2 \left(1 + \frac{|cx_n|}{|y_n|^2}\right) \leq \frac{3|y_n|^2}{2} \leq \frac{3}{2}\frac{1}{3^2}(3k)^{2^{n+1}}|x_n|^2 \leq \frac{1}{3}(3k)^{2^{n+1}}|x_{n+1}|.$$

The proof of the other inequalities are the same. This completes the proof. $\square$

As in case of dimension one, we put $\varphi_1(x,y) = \lim_{n\to\infty} x_n^{1/2^n}$ and $\varphi_2(x,y) = \lim_{n\to\infty} y_n^{1/2^n}$. If these limits exist, the map $\Phi_c(x,y) = (\varphi_1(x,y), \varphi_2(x,y))$ will give a Böttcher coordinate of F_c.

Proposition 2.1. *The map $\Phi_c(x,y) = (\varphi_1(x,y), \varphi_2(x,y))$ is holomorphic in $D_{k,R}$, depends holomorphically on c, satisfies $\Phi_c \circ F_c = F_0 \circ \Phi_c$ and*

$$\varphi_1(x,y) = x + O(1), \quad \varphi_2(x,y) = y + O(1). \tag{2.2}$$

proof. We have only to show the statements for φ_1. Since

$$x_n = \frac{x_n}{x_{n-1}^2}\frac{x_{n-1}^2}{x_{n-2}^{2^2}}\cdots\frac{x_1^{2^{n-1}}}{x^{2^n}}x^{2^n} = x^{2^n}\prod_{j=0}^{n-1}\left(\frac{x_{j+1}}{x_j^2}\right)^{2^{n-j-1}} = x^{2^n}\prod_{j=0}^{n-1}\left(1-\frac{cy_j}{x_j^2}\right)^{2^{n-j-1}},$$

it follows

$$x_n^{1/2^n} = x\prod_{j=0}^{n-1}\left(1-\frac{cy_j}{x_j^2}\right)^{1/2^{j+1}}.$$

From Lemma 2.1, the following holds under the assumption (2.1) :

$$\left|\frac{cy_j}{x_j^2}\right| \le \frac{|c|(3k)^{2^j}}{3|x_j|} \le \frac{|c|}{6}\left(\frac{3k}{R}\right)^{2^j}.$$

Then the series $\sum \frac{1}{2^{j+1}}\frac{cy_j}{x_j^2}$ converges uniformly and absolutely in $D_{k,R}$, hence the limit $\varphi_1(x,y) = \lim_{n\to\infty} x_n^{1/2^n}$ exists, holomorphic on $D_{k,R}$ and depends holomorphically on c. The functional equation follows from the definition of Φ_c.

In the proof of Lemma 2.1, we also have

$$\begin{aligned}
|x_1| &\ge \frac{|x|^2}{2} \ge R|x|,\\
|x_2| &\ge \frac{|x_1|^2}{2} \ge \frac{R^2|x|^2}{2},\\
&\cdots\\
|x_n| &\ge \frac{|x_{n-1}|^2}{2} \ge \frac{|x|^{2^n}}{2^{1+2+\cdots 2^{n-1}}} \ge \frac{(2R)^{2^n-1}|x|}{2^{2^n-1}} \ge R^{2^n-1}|x|.
\end{aligned}$$

Then it follows

$$\frac{(3k)^{2^n}}{|x_n|} \le \left(\frac{3k}{R}\right)^{2^n}\frac{R}{|x|}.$$

Hence we have

$$\log\frac{\varphi_1(x,y)}{x} = O(\frac{1}{|x|}),$$

which implies

$$\frac{\varphi_1(x,y)}{x} = 1 + O(\frac{1}{|x|}).$$

This implies (2.2). □

Corollary 2.1. *The map Φ_c extends holomorphically to the closure $\overline{D_{k,R}}$ in $\mathbb{P}^2$ and satisfies $\Phi_c|_{\overline{D_{k,R}}\cap\Pi} = id|_{\overline{D_{k,R}}\cap\Pi}$. It is injective in a neighborhood of $\overline{D_{k/2,R}}\cap\Pi$. Φ_c gives a Böttcher coordinate of F_c.*

proof. The first assertion follows from the estimate (2.2). This estimate also implies that there exists bounded holomorphic functions ξ_1 and ξ_2 on $D_{k,R}$ such that $\varphi_1(x,y) = x + \xi_1(x,y)$, $\varphi_2(x,y) = y + \xi_2(x,y)$. By Cauchy's integral formula, it follows that the jacobian matix $Jac(\Phi_c)$ satisfies

$$Jac(\Phi_c) = \begin{pmatrix} 1+\xi_{1,x} & \xi_{1,y} \\ \xi_{2,x} & 1+\xi_{2,y} \end{pmatrix} = \begin{pmatrix} 1+O(1/R) & O(1/R) \\ O(1/R) & 1+O(1/R) \end{pmatrix},$$

in $D_{k/2,R}$ for large R. Thus Φ_c is locally biholomorphic in $\overline{D_{k/2,R}}$. Especially Φ_c maps $\overline{D_{k,R}}$ onto an open neighborhood of J_Π in $\mathbb{P}^2$. Suppose Φ_c is not injective in any neighborhood of $\overline{D_{k,R}}\cap\Pi$. Then there exist sequences of distinct points $[x_n : y_n : 1]$ and $[u_n : v_n : 1]$ tending to $[x_0 : y_0 : 0]$ and $[u_0 : v_0 : 0]$ on $\overline{D_{k,R}}\cap\Pi$ respectively such that $\Phi_c(x_n, y_n) = \Phi_c(u_n, v_n)$. Then we have $\Phi_c([x_0 : y_0 : 0]) = \Phi_c([u_0 : v_0 : 0])$, hence $[x_0 : y_0 : 0] = [u_0 : v_0 : 0]$. This contradicts the local injectivity at $[x_0 : y_0 : 0]$. Thus we conclude that Φ_c gives a biholomorphism between some open neighborhoods of $J_\Pi(F_c)$ and $J_\Pi(F_0)$. The conjugacy in Proposition 2.1 implies Φ_c maps $W_0^s(J_\Pi, F_c)$ into $W_0^s(J_\Pi, F_0)$. Thus the restriction of Φ_c to $W_0^s(J_\Pi, F_c)$ is the desired Böttcher coordinate of F_c. This completes the proof. □

We can show the uniqueness of the Böttcher coordinates satisfying the property in Proposition 2.1.

Lemma 2.2. *The Böttcher coordinate of F_c, holomorphic in $D_{k,R}$ for large R satisfying (2.2), is unique.*

proof. Suppose Φ_c and Φ'_c are two Böttcher coordinates of F_c in some $\overline{D_{k,R}}$. Then the map $\Phi = \Phi'_c \circ \Phi_c^{-1}$ commutes with F_0. Put $\Phi(x,y) = (\varphi_1(x,y), \varphi_2(x,y))$. If we put $y = tx$ and $\tilde\varphi_1(x,t) = \varphi_1(x,tx)$, $\tilde\varphi_2(x,t) = \varphi_2(x,tx)$, they are holomorphic in $\{|x| > 2R,\ 1/k < |t| < k\}$ and satisfy the functional equation :

$$\tilde\varphi_j(x^2, t^2) = \tilde\varphi_j(x,t)^2, \quad (j = 1, 2).$$

We write $\tilde\varphi_1$ as

$$\tilde\varphi_1(x,t) = x + \sum_{j\ge k} \frac{a_j(t)}{x^j},$$

where a_j is holomorphic in $\{1/k < |t| < k\}$ and $k \geq 0$ is the minimum such that a_k does not identically vanish. Then, it follows

$$\begin{aligned} \tilde{\varphi}_1(x^2, t^2) &= x^2 + \frac{a_k(t^2)}{x^{2k}} + O(\frac{1}{x^{2k+2}}) \\ \tilde{\varphi}_1(x,t)^2 &= x^2 + 2\frac{a_k(t)}{x^{k-1}} + O(\frac{1}{x^k}). \end{aligned}$$

which is impossible if $k \geq 0$ is finite. Thus $\tilde{\varphi}_1(x,t) \equiv x$, hence $\varphi_1(x,y) \equiv x$. The same holds for φ_2. Then $\Phi = id$, hence $\Phi_c = \Phi'_c$. This completes the proof. $\square$

Denote the Green function of F_c by G_c.

Lemma 2.3. $G_c(x,y) = max(\log^+ |\varphi_1(x,y)|, \log^+ |\varphi_2(x,y)|) = G_0 \circ \Phi_c(x,y)$.

proof. From the proof of Proposition 2.1, the limit

$$G_c(x,y) = \lim_{n\to\infty} 2^{-n} \log^+ |F_c^n(x,y)|$$

exists and continuous on $D_{k,R}$. Suppose $\log^+ |\varphi_1(x,y)| > \log^+ |\varphi_2(x,y)|$. Then there exists a $K > 1$ such that $|x_n|^{2^{-n}} > K|y_n|^{2^{-n}}$. That is, $|x_n| > K^{2^n}|y_n|$ and hence

$$|F_c^n(x,y)| = |x_n|\sqrt{1 + \left|\frac{y_n}{x_n}\right|^2} \sim |x_n|.$$

Then $G_c(x,y) = \lim_{n\to\infty} \log^+ |x_n|^{2^{-n}} = \log^+ |\varphi_1(x,y)|$. The case $\log^+ |\varphi_1(x,y)| < \log^+ |\varphi_2(x,y)|$ is similar. Now the lemma follows from the continuity of G_c, φ_1 and φ_2. $\square$

Lemma 2.4. $W_0^s(\zeta, F_c) = \{(x,y) \in A_0; \varphi_2 = \zeta\varphi_1\}$ *for any* $\zeta \in J_\Pi$.

proof. Put $\xi(x,y) = \dfrac{\varphi_2(x,y)}{\varphi_1(x,y)}$. Then, since

$$\xi(F_c(x,y)) = \frac{\varphi_2(F_c(x,y))}{\varphi_1(F_c(x,y))} = \left(\frac{\varphi_2(x,y)}{\varphi_1(x,y)}\right)^2 = \xi(x,y)^2,$$

it follows that $\xi(F_c^n(x,y)) = \xi(x,y)^{2^n}$.

Now suppose $\varphi_2(x,y) = \zeta\varphi_1(x,y)$. Then, we have $\dfrac{\varphi_2(x_n,y_n)}{\varphi_1(x_n,y_n)} = \zeta^{2^n}$. From (2.2), it follows $|\dfrac{y_n}{x_n} - \zeta^{2^n}| \to 0$. Then $[x_n : y_n : 1]$ is close to $[1 : \zeta : 0]$ in $\mathbb{P}^2$ for any $n \geq 0$. Thus $(x,y) \in W_0^s(\zeta, F_c)$.

Next, suppose $(x,y) \in W_0^s(\zeta, F_c)$. If $|\xi(x,y)| \neq 1$, $\xi(F_c^n(x,y))$ tends to ∞ or 0, which contradicts with the assumption $(x,y) \in W_0^s(\zeta, F_c)$. Hence $\dfrac{\varphi_2(x,y)}{\varphi_1(x,y)} = \tau$ for some $\tau \in J_\Pi$. Then from the first part, we have $(x,y) \in W_0^s(\tau, F_c)$. Since $W_0^s(\tau, F_c), \tau \in J_\Pi$ are mutually disjoint, we must have $\tau = \zeta$. This completes the proof. □

Corollary 2.2. $W_0^s(J_\Pi) = \{(x,y) \in A_0; |\varphi_2(x,y)| = |\varphi_1(x,y)|\}$.

Lemma 2.5. $\Phi_c|_{W_0^s(\zeta,F_c)} : W_0^s(\zeta, F_c) \to W_0^s(\zeta, F_0)$ *is conformal for any* $\zeta \in J_\Pi$.

proof. It follows from Proposition 2.1 that $\Phi_c|_{W_0^s(\zeta,F_c)}$ is holomorphic. Its image is equal to $W_0^s(\zeta, F_0)$ by Lemma 2.4. By (2,1), $\Phi_c(x,y) = (x,y) + O(1)$. This implies its conformality. This completes the proof. □

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