

# LANG-BOMBIERI'S CONJECTURES

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**ABSTRACT.** In this article we generalize Iitaka-Viehweg conjectures for geometric varieties([Km4]) to those for arithmetic ones. They are applicable to Diophantine problems as well as geometric Diophantine problems.

## 1. INTRODUCTION

We shall prove the following conjectures proposed by Lang and Bombieri([SL]):

**Conjecture 1.** *Let  $K$  be an arithmetic field and  $X$  a variety defined over  $K$ . Assume that  $X$  be a variety of general type. Then it has no dense set of  $K$ -rational points in  $X$ .*

This is well known as Mordell's conjecture in case of curves of genus  $\geq 2$  and is shown by Faltings([F]).

There is an analogue of the conjecture, which is proposed by Noguchi([No], [Km1], [Km2], [Km3]):

**Conjecture 2.** *Let  $X$  and  $S$  be algebraic varieties over the field of the complex numbers. Assume that  $X/S$  be a fibre space with the geometric generic fibre of general type. If  $X/S$  has a dense set of rational sections in  $X$ , then  $\text{var}(X/S) = 0$ .*

First applying Kollar-Kawamata's theorem([Kaw], [Ko2], [Ko1], [Mat]), we shall prove the conjecture 2. Secondary we shall show an arithmetic version the conjecture 1 later.

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## 2. REVIEW IITAKA-VIEHWEG CONJECTURE

Iitaka proposed the following conjecture in 1970:

**Conjecture 3.** ([I]) *Let  $X/S$  be a fibre space over the field of the complex numbers and  $X_{\bar{\eta}}$  the geometric generic fibre of  $X/S$ . Then  $\kappa(X) \geq \kappa(X_{\bar{\eta}}) + \kappa(S)$ .*

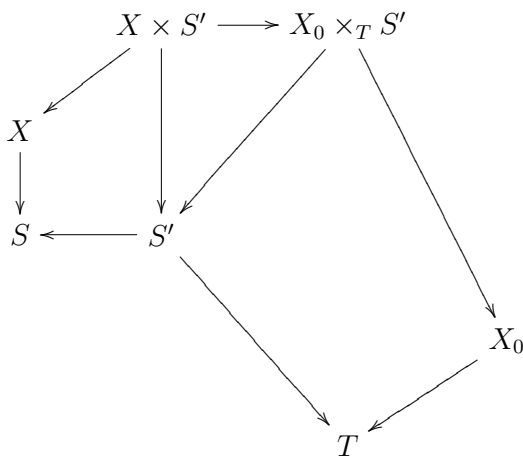
Viehweg raised the following

**Conjecture 4.** ([V]) *Let  $f : X \rightarrow S$  be a fibre space  $X/S$  with the geometric generic fibre of Kodaira dimension  $\geq 0$ . Then there exists a number  $m$  such that*

$$\kappa(\det f_* \omega_{X/S}^{\otimes m}) \geq \text{var}(X/S).$$

**Notation 1.** ([SGA], [GG], [I], [V], [Ko2], [Dix], [MiyPet]), [Mat])

- (1) Let  $k$  be a field. A geometrically irreducible, reduced, smooth scheme  $X$  over  $k$  is said to be a non singular variety over  $k$ .
- (2) Let  $X$  be a non singular variety of dimension  $d$  and  $\Omega_X^1$  the differential sheaf over  $X$ . Let  $\omega_X$  denote  $\Omega_X^d$ .
- (3) A connected proper surjective morphism  $f : X \rightarrow S$  of non singular varieties  $X$  and  $S$  is said to be a fibre space  $X/S$ .
- (4) Let  $f : X \rightarrow S$  be a fibre space  $X/S$ .  $\omega_{X/S}$  denotes  $\omega_X \otimes f^* \omega_S^{-1}$ .
- (5) Let  $L$  be an invertible sheaf over  $X$ .  $\kappa(L)$  denotes the maximal dimension of the image variety of the rational map  $X \rightarrow \mathbf{P}(\Gamma(X, L^{\otimes m}))$  defined by  $\Gamma(X, L^{\otimes m}) \otimes \mathcal{O}_X \rightarrow L^{\otimes m}$  for  $m \gg 0$ . We call  $\kappa(L)$  Iitaka dimension of  $L$ .
- (6)  $\kappa(\omega_X)$  is said to be Kodaira dimension of  $X$ , which is denoted by  $\kappa(X)$ .
- (7) Let  $X/S$  be a fibre space. We denote by  $\text{var}(X/S)$  the minimal dimension of  $T$  such that there exists a generically finite morphism  $S' \rightarrow S$  in which  $X \times_S S'$  is birationally equivalent to  $S' \times_T X_0$  for some varieties  $T$ ,  $X_0$  with  $X_0/T$  a fibre space. This dimension is called Viehweg dimension of a fibre space  $X/S$ .



- (8) *The category of bands*([Gir], [SGA]) *of profinite groups*([Se], [Shatz], [RBZL], [Zuo]) *is defined in the following. The objects are the profinite groups and the arrows are the homomorphisms of profinite groups modulo inner automorphisms.*
- (9) *A  $\mathbf{Q}$ -divisor  $D$  is said to be effective if  $\kappa(D) \geq 0$ .*

We have proved the two conjecture 3 and 4. Recall the sketch of the proof.

- (1) Let  $X/S$  be a fibre space with the geometric generic fibre of Kodaira dimension  $\geq 0$  ([I]). We may assume  $\text{var}(X/S) = \dim S$  by Viehweg's comment. Construct a generically finite cover  $Y$  of  $X$  such that
- (a)  $Y$  is a variety of general type or a variety with the abundant canonical divisor,
  - (b)  $\max_{m>0} \kappa(\det f_* \omega_{X/S}^{\otimes m}) = \max_{m>0} \kappa(\det g_* \omega_{Y/S}^{\otimes m})$ , where  $\max_{m>0} \kappa(\det g_* \omega_{Y/S}^{\otimes m}) \geq \text{var}(Y/S)$  is obtained by Kollar and Kawamata independently.

(2)

$$\text{var}(Y/S) \geq \text{var}(X/S).$$

The following Mochizuki's theorem is available to prove (2) above.

**Theorem 1.** ([Mch]) *Let  $p$  be a prime number. Let  $K$  be a subfield of a finitely generated field extension of  $\mathbb{Q}_p$ . Let  $L, M$  be function fields of arbitrary dimension over  $K$ . Let  $\text{Hom}_{\text{Spec}(K)}(\text{Spec}(L), \text{Spec}(M))$  be the set of  $K$ -morphisms from  $\text{Spec}(L)$  to  $\text{Spec}(M)$ . Let  $\text{Hom}_{\Gamma_K}^{\text{open}}(\Gamma_L, \Gamma_M)$  over  $\Gamma_K$ , the set of open continuous homomorphisms of profinite groups considered up to composition with an inner automorphism arising from  $\text{Ker}(\Gamma_M, \Gamma_K)$ , where  $\Gamma_L$  and  $\Gamma_M$  are the absolute Galois groups of  $L$  and  $M$ , respectively. Then the natural map  $\text{Hom}_{\text{Spec}(K)}(\text{Spec}(L), \text{Spec}(M)) \rightarrow \text{Hom}_{\Gamma_K}^{\text{open}}(\Gamma_L, \Gamma_M)$  is bijective.*

Let  $p$  be a prime number. Let  $K$  be a subfield of a finitely generated field extension of  $\mathbb{Q}_p$ . Note that there exists an isomorphism  $\iota : \bar{K} \cong \mathbf{C}$ .

Let  $X_{\bar{\eta}}$  be the geometric generic fibre of  $X/S$ . Then there exists a variety  $F_{K_0}$  and a finite extension field  $K_0$  of  $\mathbf{Q}$  such that  $F_{K_0} \times_{K_0} \bar{K}_0 \cong X_{\bar{\eta}}$ .

Let  $\text{Bir}_{\mathbf{C}}(X_{\bar{\eta}}) = \text{Bir}_{\bar{K}_0}(F_{K_0} \otimes_{K_0} \bar{K}_0)$ . There exists a definition field  $K$  of  $\text{Bir}_{\bar{K}_0}(F_{K_0} \otimes_{K_0} \bar{K}_0)$  such that  $\text{Bir}_K(F_K) \otimes_K \bar{K} \cong \text{Bir}_{\mathbf{C}} X_{\bar{\eta}}$  and that  $K$  is a finite extension field of  $\mathbf{Q}$  since  $\text{Bir}(X_{\bar{\eta}})$  is an algebraic group. Fix  $K$  once for all.

Let  $\pi : \Gamma_{F_K} \rightarrow \Gamma_K$  denote the structure map associated to  $F_K \rightarrow \text{Spec}(K)$ , which is a surjection since  $K$  is algebraically closed in the rational function field of  $F$ . Let  $Z(\Gamma_{F_K})$  denote the centre of  $\Gamma_{F_K}$ . Then  $\pi$  induces  $\pi : Z(\Gamma_{F_K}) \rightarrow Z(\Gamma_K)$ .

Hence consider a crossed module

$$(\pi^{-1}(Z(\Gamma_K)) \rightarrow \text{Aut}_{\Gamma_K}(F_K)).$$

There exists an exact sequence by Breen([Breen1], [Breen2], [BJ])

$$0 \rightarrow Z(\Gamma_{F_K})[1] \rightarrow (\pi^{-1}(Z(\Gamma_K)) \rightarrow \text{Aut}_{\Gamma_K}(F_K)) \rightarrow \text{Out}_{\Gamma_K}(F_K) \rightarrow 1$$

and a long exact sequence:

$$0 \rightarrow H^1(\Gamma_{S_K}, Z(\Gamma_{F_K})[1]) \rightarrow H^1(\Gamma_{S_K}, (\pi^{-1}(Z(\Gamma_K)) \rightarrow \text{Aut}_{\Gamma_K}(F_K))) \rightarrow H^1(\Gamma_{S_K}, \text{Out}_{\Gamma_K}(F_K))$$

We proved the following lemma making use of revised Matsumura's theorem.

**Lemma 1.** ([Mat]) *Let  $F_K$  be a variety of  $\kappa(F_K) \geq 0$ . Then  $\text{Bir}_K(F_K)$  has at most countable connected components and the connected component which contains an identity element is of finite type over  $K$ .*

We can take base change  $K'/K$  which is a finite extension so that the image of the extension element  $\Gamma_{X_K}$  associated to  $X/S$  is a trivial map in  $H^1(\Gamma_{S_K}, \text{Out}_{\Gamma_K}(F_K)) \cong \text{Hom}(\Gamma_{S_K}, \text{Out}_{\Gamma_K}(F_K))$ . Thus  $\Gamma_{X_K}$  is going up to  $H^1(\Gamma_{S_K}, Z(\Gamma_{F_K})[1])$ , which turns out to a central extension.

To prove  $\text{var}(Y/S) \geq \text{var}(X/S)$ , we have proved that it is enough to show that  $\text{var}(Y/S) = 0$  implies  $\text{var}(X/S) = 0$ .

Assume  $\text{var}(Y/S) = 0$ . Then the extension  $1 \rightarrow \Gamma_{F_K} \rightarrow \Gamma_{X_K} \rightarrow \Gamma_{S_K} \rightarrow 1$  associated to a fibre space  $X/S/\text{Spec } K$  splits. Hence  $\Gamma_{X_K}$  is a semi-direct product. This extension is central since the argument above. On the other hand, a central extension which is a semi-direct product is a trivial product.

By Breen's theory([Breen1], [Breen2], [AM]), we have

$$H^1(\Gamma_{S_K}, (\pi^{-1}(Z(\Gamma_K)) \rightarrow \text{Aut}_{\Gamma_K}(F_K))) \cong \text{Ext}_{\Gamma_K}(\Gamma_{S_K}, \Gamma_{F_K}),$$

whose neutral element is  $\Gamma_{S_K} \times_{\Gamma_K} \Gamma_{F_K}$ . The sketch of the proof finishes.

### 3. APPLICATION TO GEOMETRIC DIOPHANTUS PROBLEMS

**Theorem 2.** *Let  $X/S$  be a fibre space with the geometric generic fibre of general type. Assume that there is a dense set of rational sections of  $X/S$  in  $X$ . Then  $\text{var}(X/S) = 0$ .*

*Proof.* It suffices to prove the assertion for every curve  $C$  on  $S$ . Namely we will prove for a fibre space  $X_C/C$ . Change the base curve  $C$  by a normal curve  $C'$  of genus  $g(C') \geq 2$ . Let  $X/C$  denote such a fibre space and an image of a rational section  $C_\lambda$ , which is in fact a section of  $X/C$ .

Since  $\kappa(\omega_{X/C}) \geq \kappa(\omega_{X_{C'}})$ , there exist a number  $m$  and the following commutative square for  $\dim X = n$ :

$$\begin{array}{ccc}
 & \Omega_X^{\otimes mn} & \\
 & \uparrow & \nwarrow \\
 & \mathcal{O}_X(mK_X) & \\
 & \uparrow & \\
 \Omega_C^{\otimes m} & \longrightarrow & \Omega_C^{\otimes mn}
 \end{array}$$

Restricting the commutative diagram above to a curve  $C_\lambda$  on  $X$ , we have

$$\begin{array}{ccc}
 & \Omega_X^{\otimes mn}|_{C_\lambda} & \\
 & \uparrow & \nwarrow \\
 & \mathcal{O}_X(mK_X)|_{C_\lambda} & \\
 & \uparrow & \\
 \Omega_C^{\otimes m}|_{C_\lambda} & \longrightarrow & \Omega_C^{\otimes mn}|_{C_\lambda}
 \end{array}$$

Thus we have the upper bound of the intersections

$$(C_\lambda, K_X) \leq n \deg \Omega_C^1 = n(2g(C) - 2).$$

Since  $\kappa(C) = 1$  and  $\kappa(\omega_X) \geq \kappa(\omega_{X_{\bar{\eta}}}) + \kappa(\omega_C)$ , we obtain  $\kappa(\omega_X) = \dim X$ . Hence  $\omega_X$  is big.

For any ample invertible sheaf  $L$  there exists a number  $\ell$  such that  $L \hookrightarrow \omega_X^\ell$  (Kodaira's Lemma). We have the estimation of intersections  $(C_\lambda, L) \leq \ell(C_\lambda, \omega_X) \leq n\ell(2g(C) - 2)$  for every curve which is not contained in the stable base locus of  $\omega_X$ . These curves are parametrized by Hilbert schemes  $Hilb_X^{p(m)}$ , where the Hilbert polynomials  $p(m)$  are a finite number of polynomials since  $(C_\lambda, L) \leq n\ell(2g(C) - 2)$ . Hence one can find a variety  $T$  which is a component of parametrizing Hilbert schemes and a dominant rational map  $\psi$  over  $C$  such that  $\psi : T \times C \rightarrow X$  over  $C$ . See the following diagram:

$$\begin{array}{ccccc}
 T & \longleftarrow & \mathbf{\Gamma} = \{C_\lambda\} \subset T \times X & \longrightarrow & X \\
 \uparrow & & \swarrow & & \downarrow \\
 T \times C & \xrightarrow{\hspace{2cm}} & & & C
 \end{array}$$

Here  $\mathbf{\Gamma}$  is the restriction of the universal family over  $Hilb_X^{p(m)}$  to  $T$ . Hence  $\mathbf{\Gamma}$  is birationally equivalent to  $T \times C$ . Therefore

$$0 = \text{var}(T \times C/C) \geq \text{var}(X/C)$$

and  $\text{var}(X/S) = 0$ . □

**Theorem 3.** *Let  $k$  be an algebraically closed field of characteristic 0. Let  $X$  be a variety of general type over  $k$  and  $g$  a number which is not less than 1. Then the set of non singular curves  $\{C_\lambda\}$  of genus  $g$  in  $X$  which do not contained in a fixed hyperplane forms a bounded set, i.e., given an ample invertible sheaf  $L$  over  $X$ ,  $\chi(C_\lambda, L^{\otimes m})$  are a finite number of polynomials in  $m$ .*

*Proof.* Let  $X \rightarrow \mathbf{P}^1$  be a fibre space obtained by Lefschetz pencil up to birational equivalence. For each curve  $C_\lambda$  in  $X$ , we have a fibre space  $X \times_{\mathbf{P}^1} C_\lambda \rightarrow C_\lambda$ . Applying the argument above to these fibre spaces, we complete the proof.  $\square$

Note that even if  $g = 0$ , the assertion holds since it is enough to change the direction of a homomorphism in the diagram above to  $\omega_{C_\lambda}^{\otimes nm} \rightarrow \omega_{C_\lambda}$ .

Next we shall prove the following result which is an analogue of Hermite's theorem.

**Theorem 4.** *Let  $C$  be a curve,  $S$  a set of points on  $C$  and  $d$  a number. Given a covering degree  $\leq d$  of covers over  $C$ , there exists only a finite number of finite covers over  $C$  which are etale outside  $S$ .*

*Proof.* Let  $B$  be such a finite cover over  $C$ . There are a finite number of  $\omega_{B/C}$  for all such covers  $B$ . Since  $B \cong \mathbf{Proj}(\oplus_{m \geq 0} \omega_{B/C}^{\otimes m})$ , we get the theorem.  $\square$

**Theorem 5.** *Let  $X$  be a variety and  $C, D$  curves. Let  $f : X \rightarrow C \times D$  be a proper surjective morphism. Assume  $f$  is smooth over  $C^\circ \times D^\circ$ , where  $C^\circ$  and  $D^\circ$  are open subsets of  $C$  and  $D$ , respectively. Then  $\text{var}(f) \leq 1$ .*

To prove an analogue of Lang-Trotter's conjectures it seems to be applicable .

#### 4. ARITHMETIC IITAKA-VIEHWEG CONJECTURE

**Definition 1.** *Let  $K$  be a finite extension of  $\mathbf{Q}$ . Let  $\mathcal{O}_K$  denote the integral closure of  $\mathbf{Z} \subset K$ . We call an arithmetic variety if it is an irreducible and reduced scheme which is flat and projective over  $\text{Spec } \mathcal{O}_K$ :*

$$f : \mathcal{X} \rightarrow \text{Spec } \mathcal{O}_K$$

and if, moreover, the base change  $f : X \rightarrow \text{Spec } K$  is an irreducible and reduced morphism.

We propose an arithmetic Iitaka-Viehweg conjecture:

**Conjecture 5.** *Let  $f : \mathcal{X} \rightarrow \mathcal{S}$  be an arithmetic fibre space defined over  $\text{Spec } \mathcal{O}_K$  and let the geometric generic fibre of the fibre space of Kodaira dimension  $\geq 0$ .*

- (1) *If  $\dim \mathcal{S} \geq 2$ ,  $\max_{m > 0} \kappa(\det f_* \omega_{\mathcal{X}/\mathcal{S}}^{\otimes m}) \geq \text{var}(\mathcal{X}/\mathcal{S}) \geq 1$ .*
- (2) *If, especially,  $\mathcal{S} = \text{Spec } \mathcal{O}_K$ ,  $\max_{m > 0} \kappa(\det f_* \omega_{\mathcal{X}/\mathcal{S}}^{\otimes m}) \geq \text{var}(\mathcal{X}/\mathcal{S}) \geq 1$*

Note that the latter conjecture is essential. By taking pull-back through  $\text{Spec } K \rightarrow \text{Spec } \mathcal{O}_K$ , we have an ordinary fibre space  $X/S$  over  $\text{Spec } K$ . By Iitaka's lemma, we obtain  $\kappa(\det f_* \omega_{X/S}^{\otimes m}) \leq \kappa(\det f_* \omega_{X/S}) + 1$ . Note that  $\text{var}(\mathcal{X}/\mathcal{S}) \geq \text{var}(X/S) + 1$ . If the geometric generic fibre is of general type, we can show  $\max_{m>0} \kappa(\det f_* \omega_{X/S}^{\otimes m}) \geq \max_{m>0} \kappa(\det f_* \omega_{X/S}) + 1$ . In general, construct a cover such that the fibre is of general type and that the same value as  $\max_{m>0} \kappa(\det f_* \omega_{X/S}^{\otimes m})$  and we get  $\max_{m>0} \kappa(\det f_* \omega_{X/S}^{\otimes m}) \geq \max_{m>0} \kappa(\det f_* \omega_{X/S}) + 1$ .

**Lemma 2** (Moriwaki([Mo])). *Let  $K$  be an algebraic field,  $X$  a projective irreducible reduced variety over  $K$  and  $L$  an invertible sheaf over  $X$ . Then there exists a couple  $(\mathcal{X}, \mathcal{L})$  such that*

- (1)  $\mathcal{X} \rightarrow \text{Spec}(\mathcal{O}_K)$  is an arithmetic projective variety and its generic fibre is  $X$ .
- (2) Forgetting the underlying metric structure  $\mathcal{L}$ , it is  $\mathbf{Q}$ -equivalent to  $L$ .

**Lemma 3.**  $f_* \omega_{X/S}^{\otimes m}$  is weakly positive for  $m > 0$ .

*Proof.* Let  $\mathcal{L}$  be an ample invertible sheaf over  $\mathcal{S}$ . Then  $H^1(\mathcal{X}, \omega_{\mathcal{X}} \otimes f^* \mathcal{L}) \rightarrow H^1(\mathcal{X}, \omega_{\mathcal{X}} \otimes f^* \mathcal{L}^{\otimes m})$  is injective for  $m \geq 2$  since Koll'ar-Viehweg results. Hence  $H^1(\mathcal{S}, f_* \omega_{\mathcal{X}} \otimes \mathcal{L}) = 0$ . Using the dualizing sheaf, we have

$$H^0(\mathcal{S}, \omega_{\mathcal{O}_K} \otimes \underline{\text{Hom}}_{\mathcal{O}_K}(f_* \omega_{\mathcal{X}} \otimes \mathcal{L}, \mathcal{O}_K)) = 0$$

Hence  $f_* \omega_{X/S}$  is weakly positive. By Viehweg's method([V], [Fuj], [Ws]), we can complete the proof.  $\square$

**Lemma 4.** Assume  $\kappa(\omega_X) = \dim X$  and  $\dim \mathcal{S} = 1$ . Then  $\deg \det f_* \omega_{X/S}^{\otimes m} > 0$ .

*Proof.* By weak positivity,  $\deg \det f_* \omega_{X/S}^{\otimes m} \geq 0$ . If  $\deg \det f_* \omega_{X/S}^{\otimes m} = 0$ , then  $\text{var}(\mathcal{X}/\mathcal{S}) = 0$ , which is absurd.  $\square$

**Proposition 1.** (1) Assume  $\kappa(\omega_X) = \dim X$  and  $\dim \mathcal{S} = 1$ . Then  $\max_{m>0} \kappa(\det f_* \omega_{X/S}^{\otimes m}) \geq \text{var}(\mathcal{X}/\mathcal{S}) \geq 1$ .  
 (2) Assume  $\kappa(\omega_X) = \dim X$ . Then  $\max_{m>0} \kappa(\det f_* \omega_{X/S}^{\otimes m}) \geq \text{var}(\mathcal{X}/\mathcal{S}) \geq 1$ .

*Proof.* Take a generically finite cover  $\mathcal{Y}$  such that

- (1)  $\max_{m>0} \kappa(\det f_* \omega_{Y/S}^{\otimes m}) = \max_{m>0} \kappa(\det f_* \omega_{X/S}^{\otimes m})$ .
- (2)  $\kappa(Y) = \dim Y$ ,  $\max_{m>0} \kappa(\det f_* \omega_{Y/S}^{\otimes m}) \geq \text{var}(\mathcal{Y}/\mathcal{S}) \geq 1$

Making use of  $\text{var}(\mathcal{Y}/\mathcal{S}) \geq \text{var}(\mathcal{X}/\mathcal{S})$ , we obtain the theorem.  $\square$

## 5. APPLICATION OF ARITHMETIC IITAKA-VIEHWEG CONJECTURE

**Theorem 6.** Let  $X/S$  be a fibre space with  $X$  of general type. Then  $X$  has no dense set of rational points.

In other words, we have the following theorem.

**Theorem 7.** *Let  $\mathcal{X}/\mathcal{S}$  be an arithmetic fibre space with the generic fibre  $X$  of general type. Then there exists no dense set of rational sections.*

*Proof.* We will apply the following theorem.

**Lemma 5** (Northcott([Szp]), [Mo]). *For any  $\epsilon > 0$  and  $M$  the set  $\{x \in \mathbf{P}^n(\bar{\mathbf{Q}}) | [\mathbf{Q}(x) : \mathbf{Q}] \leq e, h_{nv}(x) \leq M\}$  is finite.*

By Iitaka's conjecture,  $\kappa(\omega_{\mathcal{X}/\mathcal{S}}) \geq 0$ . Take the base change such that  $[K : \mathbf{Q}] > 1$ . We have

$$0 \rightarrow \mathcal{O}_K \rightarrow \omega_{\mathcal{O}_K} \rightarrow \Omega_{\mathcal{O}_K/\mathbf{Z}} \rightarrow 0.$$

$$\begin{array}{ccc} & \Omega_{\mathcal{X}}^{\otimes mn} & \\ \uparrow & \nearrow & \\ \mathcal{O}_X(mK_X) & & \\ \uparrow & & \\ \omega_S^{\otimes m} & \longrightarrow & \Omega_{\mathcal{S}/\mathbf{Z}}^{\otimes mn} \end{array}$$

Restrict the commutative diagram above to a curve  $\mathcal{S}_\lambda$  on  $X$ , we have

$$\begin{array}{ccc} & \Omega_{\mathcal{X}}^{\otimes mn}|_{\mathcal{S}_\lambda} & \\ \uparrow & \nearrow & \\ \omega_{\mathcal{X}}^{\otimes m}|_{\mathcal{S}_\lambda} & & \\ \uparrow & & \\ \omega_S^{\otimes m}|_{\mathcal{S}_\lambda} & \longrightarrow & \Omega_{\mathcal{S}_\lambda/\mathbf{Z}}^{\otimes mn} \end{array}$$

**Proposition 2.** *We have the following results:*

- (1) Let  $K_\ell$  be the kernel such that the following becomes exact sequences :  $0 \rightarrow K_\ell \rightarrow$

$$\omega_{\mathcal{O}_K}^{\otimes n\ell} \rightarrow \Omega_{\mathcal{O}_K/\mathbf{Z}}^{\otimes n\ell} \rightarrow 0.$$

- (2) We have an injection  $\mathcal{O}_K \rightarrow K_\ell$  and

- (3)

$$H^1(\mathcal{S}_\lambda, \underline{\text{Hom}}(\omega_{\mathcal{X}/\mathbf{Z}}|_{\mathcal{S}_\lambda}, K_\ell)) \cong H^0(\mathcal{S}_\lambda, \omega_{\mathcal{O}_K} \otimes \omega_{\mathcal{X}/\mathbf{Z}}|_{\mathcal{S}_\lambda} \otimes K_\ell^{-1}) = 0$$

for  $\ell \gg 0$ .

Let  $\Phi_K$  denote the set of the places of  $K$  at infinities and  $r_1$  the number of the real places,  $r_2$  the number of the complex places  $\sigma$  such that  $\sigma$  or  $\bar{\sigma}$  belongs to  $\Phi_K$  ([Szp]).

Note that  $\deg \omega_{\mathcal{O}_K} = -2\xi(\mathcal{O}_K) = -2\log(2^{r_2}D^{-1/2})([\text{Szp}])$ . Apply the following lemma to get the proof.



**Lemma 6.** *Let  $\cap_n \text{Supp}(\text{Coker}(H^0(C, nL) \otimes \mathcal{O}_C \rightarrow nL)) = \text{SBs}(L)$ . Then there exists a number  $N$  such that for any point  $x \in (X \setminus \text{SBs}(L))(\bar{K})$ ,  $h_{(C, \mathcal{L})}(x) \geq N$ .*

**Lemma 7** (Northcott([Szp])). *Let  $X$  be a projective variety over  $\text{Spec } K$  and  $L$  an ample invertible sheaf over  $X$ . Let  $h_L : X(\bar{K}) \rightarrow \mathbf{R}$  denote the height function associated to  $L$ . Then for any positive real numbers  $\epsilon$ ,  $M$ , the set  $\{x \in X(\bar{K}) \mid [K(x) : \mathbf{Q}] \leq \epsilon, h_L(x) \leq M\}$  is a finite set.*

This completes the theorem. □

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