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NON COMMUTATIVE ALGEBRAIC AND BIRATIONAL GEOMETRY

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Contents

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1. Non Commutative Algebraic Geometry

When he worked on creating an extensive catalog of nebulae, Sir William Herschel announced its discovery on March 13, 1781, expanding the known boundaries of the solar system for the first time in modern history. In his later career, Herschel discovered two moons of Saturn, Mimas and Enceladus; as well as two moons of Uranus, Titania and Oberon.

In commutative algebraic geometry, spaces are understood as sheaves of sets on the category of affine schemes with fpqc topology.

In noncommutative algebraic geometry, topologies are replaced by **Q**-categories and stacks are sheaves of categories on a Q-category. We introduce non commutative algebraic geometry after M. Kontsevich and A. Rosenberg. Given a class of coverings τ on the category of stacks over a **Q**-category, τ-locally affine stacks are defined to be locally representable stacks.

With respect to smooth, étale and Zariski quasi-topologies on the category of stacks over the category **Aff** of noncommutative affine schemes endowed with the fpqc quasi-topology, locally affine stacks are Artin, Deligne-Mumford and Zariski noncommutative version stacks, respectively.

Owing to local construction technic, we have affine and projective vector bundles correspondoing to a quasi-coherent module on locally affine stacks and Grassmannians corresponding a couple of locally projective quasi-coherent modules on locally affine stacks.

Every quasi-topology on a category A induces a structure of a **Q**-category on the opposite category A^o. A fibred category over the category A is a stack (resp. prestack) if the quasi-topology is coarser than that of 2-descent (resp. 1-descent).

An object of a **Q**-category is a pair of functors $\bar{A} \underset{u^*}{\overset{u_*}{\rightleftarrows}} A$ such that the functor u^* is fully faithful and left adjoint to u_*, which implies that A is a quotient category of $\bar{A}$ and u_* is a localization functor. A morphism from a **Q**-category $\bar{A} \underset{u^*}{\overset{u_*}{\rightleftarrows}} A$ to a **Q**-category $\bar{A}' \underset{u^*}{\overset{u_*}{\rightleftarrows}} A'$ is a triple $(\Phi, \bar{\Phi}, \phi)$, where $\Phi : A \to A'$ and $\bar{\Phi} : \bar{A} \to \bar{A}'$ and ϕ is a functor isomorphism $\Phi\, u_* \to u'_* \bar{\Phi}$. When $\bar{A}^o \underset{u^{*o}}{\overset{u_*^o}{\leftrightarrows}} A^o$ is a **Q**-category, $\bar{A} \underset{u^*}{\overset{u_*}{\rightleftarrows}} A$ is said to be a $\mathbf{Q}^o$-category. Let A be a category and $\mathbf{S}A$ denote the category of cosieves on A defined in the following. The objects of $\mathbf{S}A$ are couples (x, R), where x is an object of A and R is a cosieve in $x\backslash A$. Morphisms from (x, R) to (x', R') are morphisms$x \to x'$ with $R'_f \subset R$, where R'_f is a cosieve in $x \setminus A$ the objects of which are all couples $(v, \xi \circ f)$ such that $(v, \xi) \in \mathrm{Ob}\, R'$. There is a functor $u^* : A \to \mathbf{S}A$ which associates to every object x of A the couple $(x, x \setminus A)$. The functor u^* is fully faithful and the canonical right adjoint,

$$u_* : \mathbf{S}A \to A, \quad (x, R) \mapsto x$$

, which defines a $\mathbf{Q}$-category of cosieves, $\mathbf{S}A \rightleftarrows A$. Cosieves in $x \setminus A$ correspond biunivoquely to subfunctors of the functor $A(x, -)$. Morphisms from (x, R) to (y, S) are morphisms $x \xrightarrow{f} y$ such that a morphism $S \to R$ of the subfuctors is induced by $A(f, -) : A(y-) \to A(x, -)$. The functor u^* is left adjoint to u_* and is given by $u' : x \in A \to (x, A(x, -))$.

Let A be a category and $\mathbf{T}$ a map which gives each $x \in A$ a set $\mathbf{T}(x)$ of subfunctors of $A(x, -)$ which contains $A(x, -)$. The couple $(A, \mathbf{T})$ defines the full $\mathbf{Q}$-subcategory $\bar{A} \rightleftarrows A$ of the $\mathbf{Q}$-category $\mathbf{S}A \rightleftarrows A$ of cosieves whose objects are all couples (x, R), where $x \in \mathrm{Ob}A$ and $R \in \mathbf{T}(x)$. $\mathbf{T}$ and $(A, \mathbf{T})$ are said to be a quasi-cotopology and a quasi-cosite, respectively, if the next two conditions are satisfied:

- **(a):** for R, $R' \in \mathbf{T}(x)$, $R \cap R' \in \mathbf{T}(x)$,
- **(b):** if $R \in \mathbf{T}(x)$ and R' is a subfunctor of $A(x, -)$ which contains R, then $R' \in \mathbf{T}(x)$.

Quasi-topology and quasi-site on a category A are defined by quasi-cotopology and quasi-cosite on the opposite category A^o, respectively. A site $(A, \mathbf{T})$ is a stronger condition than a quasi-site, i.e.,

- **(i):** for $R \in \mathbf{T}(x)$ and $f : y \to x$, $R_f = R \times_{A(-,x)} A(-y)$ belongs to $\mathbf{T}(x)$,
- **(ii):** $R \in \mathbf{T}(x)$ and R' is a subfunctor of $A(-, x)$ such that $R'_f \in \mathbf{T}(y)$ for every $f \in R(y)$, $y \in \mathrm{Ob}A$, then $R' \in \mathbf{T}(x)$.

Cosites and cotopology are the dual notion of sites and topology, respectively. The $\mathbf{Q}$-category of cosieves $(\mathbf{S}A \rightleftarrows A)$ is the finest and the $(\bar{A}_{dis} \rightleftarrows A)$, where A_{dis} is given by all couples $(x, x \setminus A)$, $x \in \mathrm{Ob}A$.

Given a $\mathbf{Q}$-category $\bar{A} \underset{u^*}{\overset{u_*}{\rightleftarrows}} A$, we define the comma category $(id_{\bar{A}}, u^*)$ whose objects are triples (y, f, x) such that there is a morphism $f : id(y) \to u^*(x)$, where $y \in \bar{A}$, $x \in A$ and morphisms from (y, f, x) to (y^*, f^*, x^*) are couples (a, b) such that $u^*(b) \circ f = f^* \circ id(a)$.

A functor Φ from $\bar{A}$ to $\mathbf{S}A$ is defined by the assignment $y \in \bar{A}$ to the couple $(u_*(y), R_y)$, where R_y denotes the cosieve in $u_* \setminus A$ formed by the $(v, u_*(y) \xrightarrow{\xi} v)$ such that $\xi = \eta_u^{-1}(v) \circ \bar{\xi}$ for a $\bar{\xi} : y \to u^*(v)$. Note that $\eta_u : u_* \circ u^* \cong id_A$ is an isomorphism, since u^* is full and faithful.

There is a **Q**-categorical morphism from $\bar{A} \overset{u_*}{\underset{u^*}{\rightleftarrows}} A$ to $(\mathbf{S}A \rightleftarrows A)$. The quasi-cosite $(\mathbf{T}_A, A)$ associated to $\bar{A} \rightleftarrows A$ is defined to be the smallest quasi-cosite containing the image functor Φ.

Let A be a category and τ a function which gives an object $x \in A$ a family τ_x of sets of arrows to x in A/x containing id_x. The category A_τ is defined to be a category whose objects are couples $(x,\mathcal{U})$, where $x \in \mathrm{Ob}A$, $\mathcal{U} \in \tau_x$ and whose morphisms from $(x,\mathcal{U})$ to $(y,\mathcal{V})$ are morphisms such that $x_u \to x$ in $\mathcal{U}$ factors through an arrow in $\mathcal{V}$. The functor from A_τ to A which assigns to $(x,\mathcal{U})$ an object x and to $f : (x,\mathcal{U}) \to (y,\mathcal{V})$ a morphism $f : x \to y$ is right adjoint to the fully faithful functor $A \to A_\tau$ which assigns to x $(x, \{id_x\})$. This defines a $\mathbf{Q}^o$-category $(A_\tau \rightleftarrows A)$.

The function τ is called quasi-pretopology if

(1) if $\{x_u \to x| \in \mathcal{U}\}$ is a cover and $\{x_u^v \to x_u| \in \mathcal{U}_{x_u}\}$ is a cover for any u then the set of composition $\{x_u^v \to x\}$ is acover,

(2) Given two covers $\mathcal{U}, \mathcal{U}'$ of x, there exists a cover $\mathcal{U}''$ such that there exist morphisms $\mathcal{U}'' \to \mathcal{U}$ and $\mathcal{U}'' \to \mathcal{U}'$.

The dual notions of quasi-pretopology and covers are quasi- precotopology and cocovers, respectively.

Example 1. *Let k be a commutative field. Let A be the category of associative unitary k-algebras. A finite set of k-algebra morphisms $\{\phi_i : R \to R_i | i \in J\}$ is called an fpqc-cocover of R if $\{R_i \otimes_R (-)| i \in J\}$ is coservative family of exact functors. The class of fpqc-cocovers defines a quasi-pretopology on the category of affine noncommutative k-schemes.*

Definition 1. *Given a **Q**-category $\bar{A} \rightleftarrows A$, an object x in A is said to be $\bar{A} \rightleftarrows A$-sheaf if*

$$\bar{A}(y, u^*(x)) \longrightarrow A(u_*(y), x), \quad f \mapsto \eta_x^{-1} \circ u_*(f)$$

is an isomorphism for any y in $\bar{A}$.

Given a **Q**-category $\bar{A} \overset{u_*}{\underset{u^*}{\rightleftarrows}} A$ to $(\mathbf{S}A \rightleftarrows A)$ and a category $\mathcal{F}$ over A^o, we define the category $y \setminus u^*$, the category $u_*(y) \setminus A$ and the A-functor for an object y of $\bar{A}$,

$$\theta_y : y \setminus u^* \longrightarrow u_*(y) \setminus A, \quad (z, y \xrightarrow{g} u^*(z)) \mapsto (z, \eta_u^{-1}(z) \circ u_*(g))$$

, where $\eta_u : Id_A \cong u_* u^*$. The morphisms A-categories $u_*(y) \setminus A$ and $y \setminus u^*$ are cartesian and cocartesian, respectively.

$$\mathrm{Cart}_{A^o}((u_*(y) \setminus A)^o, \mathcal{F}) \longrightarrow \mathrm{Cart}_{A^o}((y \setminus u^*)^o, \mathcal{F}), \quad \Phi \mapsto \Phi \circ \theta_y$$

The A-categories $u_*(y) \setminus A$ and $y \setminus u^*$ are fibred categories over A^o.

Definition 2. *A fibred category $\mathcal{F}$ over $(A^o$ is a $\mathbf{Q}$-category $(\bar{A} \rightleftarrows A)$-stack if for any $y \in \bar{A}$,*

$$\mathrm{Cart}_{A^o}((u_*(y) \setminus A)^o, \mathcal{F}) \longrightarrow \mathrm{Cart}_{A^o}((y \setminus u^*)^o, \mathcal{F}), \quad \Phi \mapsto \Phi \circ \theta_y$$

is an equivalence.

2. Non commutative algebraic space

In this section we consider the corings that have a grouplike element g which are related to ring extensions $B \to A$. Throughout this section C denotes an A-coring. Galois corings are isomorphic to the Sweedler coring associated to a ring extension $B \to A$ induced by the existence of a grouplike element. The following theorem determines when the g-coinvariants functor is an equivalence.

Theorem 1. *Let g be a grouplike element of C, $B = A_g^{coC}$, and $G_g : M^C \to M_B$ $M \mapsto M_g^{coC}$ the g-coinvariants functor.*

(1) *The following statements are equivalent:*
  - **(i):** *(C, g) is a Galois coring and A is a flat left B-module.*
  - **(ii):** *${}_AC$ is flat and A_g is a generator in M^C.*

(2) *The following statements are equivalent, too.*
  - **(i):** *(C, g) is a Galois coring and ${}_BA$ is faithfully flat.*
  - **(ii):** *${}_AC$ is flat and A_g is a projective generator in M^C.*
  - **(iii):** *${}_AC$ is flat and $\mathrm{Hom^C(A_g, -) : M^C \to M_B}$ is an equivalence whose inverse is $- \otimes_B A : M_B \to M^C$.*

The theorem above is a restatement of one of the main results in non commutative descent theory. In fact, for an algebra extension $B \to A$, there exists a comparison functor $- \otimes_B A : M_B \to \mathrm{Desc(A/B)}$ which to each right B-module M gives a descent datum $(M \otimes_B A, f)$ with $f : M \otimes_B A \to M \otimes_B A \otimes_B A, m \otimes a \mapsto m \otimes 1_A \otimes a$. If (C, g) is a Galois coring, then the category of right C-comodules is isomorphic to the category of descent data $\mathrm{Desc(A/B)}$. Thus if $B \to A$ is faithfully flat, then it is an effective descent morphism. Furthermore, Galois corings correspond to comparison functors that are equivalences. Note that if $B \to A$ is a faithful flat extension, then $(A \otimes_B A, 1_A \otimes_B 1_A)$ is a Galois coring. The objects in the category of corings are pairs $(C : A)$, where A is an R-algebra and C is an A-coring. A morphism between corings $(C : A)$ and $(D : B)$ is a pair of mappings $(\gamma : \alpha) : (C : A) \to (D : B)$ satisfying

(1) $\alpha : A \to B$ is an algebra map. Hence D is considered to be an (A, A)-bimodule.

(2) $\gamma : C \to D$ is a map of (A, A)-bimodules such that

$$\xi \circ (\gamma \otimes_A \gamma) \circ \underline{\Delta}_C = \underline{\Delta}_C \circ \gamma, \underline{\varepsilon}_D \circ \gamma = \alpha \circ \underline{\varepsilon}_C,$$

where $\xi : D \otimes_A D \to D \otimes_B D$ is the canonical map of (A, A)-bimodules.

Since an algebra A can be considered as a trivial A-coring $(A : A)$, this category of corings contains the category of R-algebras.

Left C-comodule is defined as a left A-module M, with a coassociative and counital left C-coaction.C-morphisms between left C-comodules M, N are defined in an obvious way. Left C-comodules and their morphisms form a pre-additive category ${}^C M$.

3. Geometric View

Let k be a commutative field and A, B k-algebras. The objects of the opposite category of corings denote $\mathrm{Spec}\,(C : A)$ and a morphism between $\mathrm{Spec}\,(D : B) \to \mathrm{Spec}\,(C : A)$ denotes $\mathrm{Spec}\,(\gamma : \alpha)$. This category is said to be that of covers. Furthermore, the category ${}^C M$ is abelian and it is denoted $QCoh(\mathrm{Spec}\,(C : A))$. The canonical morphism $f : \mathrm{Spec}\,(B \otimes_A B : B) \to \mathrm{Spec}\,(A : A)$ defines an equivalence between abelian categories $f^* : QCoh(\mathrm{Spec}\,(A : A)) \cong QCoh(\mathrm{Spec}\,(B \otimes_A B : B))$. Owing to Morita-Takeuchi theorems or Grothendieck ideas, the geometry of covers consist in $QCoh(\mathrm{Spec}\,(C : A))$.

The cover $\mathrm{Spec}\,(C : A)$ equipped with an epimorphism $A \otimes A \to C$ which is a morphism of coalgebras is said to be a space cover. A morphism in the category of space covers is defined to be a morphism as covers compatible with additional structure as space covers. Let $f = (\gamma, \alpha)$, $g = (\delta, \beta)$ be two morphisms between space covers $\mathrm{Spec}\,(C : A) \to \mathrm{Spec}\,(D : B)$. When for $x_i \otimes y_i \in \ker(A \otimes A \to C$, the following equation holds $\sum_i \alpha(x_i) \cdot \beta(y_i) - \beta(x_i) \cdot \alpha(y_i) = 0$ in B, two morphisms f and g are defined to be equivalent.

Definition 3. *The category of non commutative algebraic spaces over k is the localization category with the canonical morphisms invertible of the quotient of the category of space covers by equivalence of equivalent morphisms.*

The category of separated quasi-compact schemes over k and the opposite category of that of k-algebras are equivalent to a full subcategory of the category of non commutative algebraic spaces over k, respectively. The category of non commutative algebraic spaces over k admits finite limits. A non commutative algebraic space of the type $\mathrm{Spec}\,(A : A)$, where A is a k-algebra, is said to be an affine space. Let $N\mathbf{P}_k^{d-1}$ be the non commutative projective space over k and A a

k-algebra. The set $\mathrm{Hom}(\mathrm{Spec}\,(A:A), N\mathbf{P}_k^{d-1})$ is the set of quotient modules of A^d which are locally free A-modules of dimension 1 in flat topology. In the same way, we have the non commutative Grassmannian $NGr_k(r,d)$.

4. Extension of skew fields and Galois theory

Let A be an integral domain such that $xA \cap yA \neq 0$ for $x, y \in A$, which is called a right Ore domain. Let $S = R^{\times}$. Then the localization of A at S is a skew field $K = A_S$ and the natural homomorphism $\lambda : A \to K$ is a monomorphism. Racall that every ring with a homomorphism to a field has invariant basis number and K is uncountable. From now on, we treat a non commutative algebraic space of the type $\mathrm{Spec}\,(C:A)$ where A is a Ore domain. Any equation of degree $n > 0$, $x^n + a_1x^{n-1} + \cdots + a_n = 0$ ($a_i \in K$), has a right root in some extension of K. There exists the right algebraic closure $\overline{K}$ over K such that any equation of the type above has a right root in $\overline{K}$. A Galois extension L/K is outer if and only if the centralizer of K in L is just the centralizer of L. Let k be a commutative field of characteristic 0 and K a k-algebra of finite type, skew field. Let $\overline{K}$ be the right algebraic closure of K such that the centralizer of K in $\overline{K}$ is just the centralizer of $\overline{K}$([Cohn]). Let $(K_i)_{i \in I}$ be a family of skew fields such that

(1) K_i are subfields of K,
(2) K_i are k-algebras of finite type,
(3) the centralizers of K_i in $\overline{K}$ are the center of $\overline{K}$.

Then the $\overline{K}/K_i$ are all outer Galois extensions, whose Galois groups are profinite groups. We need Jacobson-Bourbaki correspondence: Let K be a field and $End(K)$ the endomorphism ring of the additive group K^+ with the finite topology. We have an order-reversing bijection between the subfields D of K and the closed K-subrings of the type $End_{D^-}(K)$ of $End(K)$. From this, we have the following Galois connection: Let L/K be an algebraic Galois extension with Galois group G outer. Then we have a bijection between intermediate fields D, i.e., $K \subset D \subset L$ and the closed subgroups H.

5. Non commutative algebraic birational geometry

We investigate the non commutative algebraic birational geometry from the point of view of the profinite Galois groups. Let $X \to S$ be a non commutative fibre space of algebraic spaces over $\mathrm{Spec}\,(k)$, with the generic point of the generic general fibre one of skew fields K_i which are defined in the preceding section. Let $1 \to G \to E \to P \to 1$ be an extension of a profinite group P by a profinite group G associated to the non commutative fibre space $X \to S$. Hence G is a profinite group, that is one of the Galois group $Gal(\overline{K}/K_i)$. To an exact sequence $1 \to \mathrm{InnG} \to \mathrm{Aut}G \to \mathrm{Out}G \to 1$, we have an exact sequence

$$\mathrm{H}^1(P, \mathrm{Inn}G) \to \mathrm{H}^1(P, \mathrm{Aut}G) \to \mathrm{H}^1(P, \mathrm{Out}G),$$

i.e.,

$$\mathrm{Hom}(P, \mathrm{Inn}G) \to \mathrm{Hom}(P, \mathrm{Aut}G) \to \mathrm{Hom}(P, \mathrm{Out}G).$$

Here $\mathrm{Out}G$ denotes the outer automorphism group of G. A group extension is an element of $\mathrm{H}^1(P, G \to \mathrm{Aut}G)$, where $G \to \mathrm{Aut}G$ is a crossed module. We have

$$1 \to \mathrm{H}^2(P, \mathrm{Z}(G)) \to \mathrm{H}^1(P, G \to \mathrm{Aut}G) \to \mathrm{H}^1(P, \mathrm{Out}G).$$

Here $\mathrm{Z}(G)$ denotes the center of G. Assume that Out(G) is an algebraic group of countable connected components. Then the canonical representation $\rho : P \to \mathrm{Out}G$ turns out to be trivial after replacing a profinite group associated to a finite morphism $S' \to S$ in the following lemma. Furthermore assume that the extension is neutral. This assumption is satisfied since there exists a homomorphism from $1 \to G' \to G' \times P \to P \to 1$ to $1 \to G \to E \to P \to 1$, where $P \to P$ is an identity, $G' = Gal(\overline{K}/K)$.

Since we have $\mathrm{H}^2(P, \mathrm{Z}(G)) \to \mathrm{H}^1(P, G \to \mathrm{Aut}(G))$, the extension $1 \to G \to E \to P \to 1$ is given by pushing out an extension $1 \to \mathrm{Z}(G) \to E' \to P \to 1$. Hence E' is a semi-direct product $\mathrm{Z}(G) \rtimes P$, which is contained in a semi-direct product $G \rtimes P$. Thus this central extension is trivial. Therefore by pushing out this central extension, the extension $1 \to G \to E \to P \to 1$ is trivial.

Lemma 1. *There exists a homomorphism $P' \to P$ with $(P' : P) < \infty$ such that the representation $\rho : P' \to \mathrm{Out}(G)$ is trivial. Here P' denotes the absolute Galois group $\mathrm{Gal}(\overline{R(S')}/R(S'))$.*

Proof. Let A denote $\mathrm{Out}(G)$. An algebraic group A is locally algebraic. The natural representation $\rho : P \to A$ induces $\overline{\rho} : P \to A/A^0$, where A^0 denotes the neutral component of A. There is no countable profinite group. Since A/A^0 is a countable set, $\overline{\rho}(P)$ is a finite group. Replace by P the kernel of $\overline{\rho}$. We have $\rho : P \to A^0$. Hence we have an isomorphism

$$H^1(\overline{R(S)}/R(S), A^0(\overline{R(S)}) \cong H^1(BP, A^0).$$

Let P be an A^0-torsor associated to $\rho : P \to A^0$. A^0 is algebraic (quasi-compact, faithfully flat and of finite type) over $\mathrm{Spec}\,(\mathrm{R}(S))$. Thus there exists a generically finite $S' \to S$ such that an A^0-torsor P is trivial over $\mathrm{Spec}\,(\mathrm{R}(S'))$. Hence the representation $\rho : P' \to \mathrm{Out}(G)$ is trivial. □

Thus we obtain the following result in our proof.

Theorem 2. *Let $1 \to G \to E \to P \to 1$ be an extension of a profinite group P by a profinite group G. Assume*

: *(a)* $\mathrm{Out}(G)$*, is an algebraic group with countable connected components.*
: *(b) $E \to P$ has a section which is a group homomorphism, i.e., a neutral extension.*

Then there exists a profinite group P' such that the pull-back of the extension $1 \to G \times_P P' \to E \times_P P' \to P' \to 1$ is a direct product.

Let X be a non commutative fibre space of smooth varieties over $\mathrm{Spec}\, k$. We have the canonical homomorphism $\Gamma(X, \Omega_X^{\otimes m}) \otimes \mathcal{O}_X \to \Omega_X^{\otimes m}$. Assume this homomorphism is generically epimorphism. Then it determines a map from an open of X to a Grassmannian. When this map is birational, i.e., the field defined by the generic point of X and that of the image are isomorphic, the assumption (a) above is satisfied.

Remark 1. Let $\phi : G_1 \to G_2$ be an open continuous homomorphism of profinite groups. $\phi(G_1) \subset G_2$. Let $Z(G_2)C_{\phi(G_1)}(\phi(G_1))$ denote C. Then for a homomorphism between extensions of P by G_1 and G_2 respectively, one has homomorphisms $H^2(P, Z(G_1)) \to H^2(P, \phi(Z(G_1))) \to H^2(P, C)$. There exists an open subgroup P' of finite index of P such that $H^2(P', Z(G_2)) \to H^2(P', C)$ is injective.

6. Diophantine Geometry

Let X/C be a fibre space of non singular projective varieties over an algebraically closed field of characteristic 0.

Theorem 3. *Let X/C be a fibre space of non singular projective varieties over an algebraically closed field of characteristic* 0*. Assume that X has a dense set of sections in X and that C is a curve of genus $g \geq 2$. Assume the generic general fibre is of general type. Then X is isotrivial.*

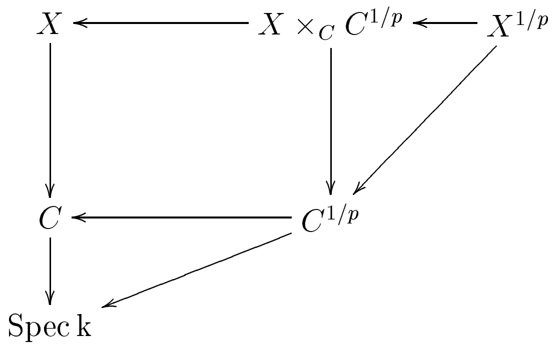
Lemma 2. *Let X/C be a fibre space of non singular projective varieties over an algebraically closed field of characteristic* 0*. Assume that X has a dense set of sections in X and that C is a curve of genus $g \geq 2$. Assume the generic general fibre is of Kodaira dimension ≥ 0. Then the intersection numbers (K_X, C) of a canonical divisor K_X and sections C except for sections contained in a hypersurface are bounded.*

We use Mori-Miyaoka-Kollar arguments.

Lemma 3. *Let k be an algebraically closed field of characteristic $p > 0$. Let X/C be a fibre space of non singular projective varieties over*

k. Assume that X is of general type and that C is a curve of genus $g \geq 2$.Assume C^{1/p^m} is a curve of genus g. Assume that X has a set of sections which is dense in X. Then intersection numbers (K_X, C_λ) with a canonical divisor K_X and sections C_λ except for sections contained in a hypersurface are bounded.

Proof. Let $C^{1/p} \to C$ be the Frobenius map.



We make use of the chow scheme considering a curve as a cycle of curves instead of making use of the Hilbert scheme. Let $Z = p^m C^{1/p^m}$ be a 1-cycle on X. By the formula of $\dim \mathrm{Chow}_Z(X) \geq -(K_X, p^m C^{1/p^m}) + p^m (\dim X)\chi(C^{1/p^m}, \mathcal{O}) = -(K_X, C) + p^m \dim X(2g-2)$ C deforms toward all direction.

Any cycle which corresponds to a k-point $\mathrm{Chow_Z(X)}$ that contains a fixed point except for C contained in a hypersurface cannot break itself since a variety X is of general type cannot contain a dense set of elliptic curves or rational curves. Hence any section except sections contained in the hypersurface moves toward all directions. If X/C, moreover, is liftable to characteristic 0, X/C is isotrivial. We have (C_λ, K_X). □

Remark 2. Let S/C be an elliptic surface with infinite sections sucth that S/C is not isotrivial. Let C be a curve of genus ≤ 2. In this case, the formula

$\dim \mathrm{Chow}_Z(X) \geq -(K_S, p^m C^{1/p^m}) + p^m(\dim S)\chi(C^{1/p^m}, \mathcal{O})$
$= -(K_S, C) + p^m \dim S(2g-2)$ implies that each section C_λ moves toward all direction as a 1-cycle, but it is deformed to become a cycle which consists of elliptic curves or rational curves.

7. Estimation of Birational Deformations for varieties of non negative Kodaira dimension

Let Y be a scheme over a scheme T. The fibred category $\mathrm{Fib_{1,Y/T}}$ over $U \in$ ob(Comm.Aff/T) has $\mathcal{O}_{Y \times_T U}$-Modules which are locally free of rank one and $\mathcal{O}_U$-flat as the objects and isomorpisms between such $\mathcal{O}_{Y\times_T U}$-Modules as morphisms.

Let k be a field of complex numbers. Let X be a projective non singular variety. Let ω_X be the canonical invertible sheaf over X. We have a morphism $\phi : X \to \mathrm{Fib}_1$. Here Fib_1 denotes $\mathrm{Fib}_{1,\mathrm{Y/T}}$, where $Y = T = \mathrm{Spec\,k}$. The image of the morphism ϕ implies the amplitude of ω_X over X. We define an analogue of Kodaira dimension by the dimension of $\phi(X)$. Let $\kappa_1(\omega_{X/S}$ be the dimension of the image $\omega_{X/S}$ onto Fib_1. Then $\kappa_1(\omega_{X/S}) \geq \kappa_1(\omega_{X_{\bar\eta}})$, $\kappa_1(\omega_{X/S}) \geq \mathrm{var(X/S)}$.

8. Birational automorphism groups in Commutative Algebraic Geometry

Theorem 4. *Let X be a non singular projective variety of Kodaira dimension ≥ 0. $Bir(X)$ is a scheme which is locally of finite type.*

Lemma 4. *Let X be a quasi-projective variety.* $\mathrm{Aut(X)}$ *is a space (resp. a stack) over the category of commutative algebras, i.e., a sheaf of sets. Furthermore,* $\mathrm{Aut(X)}$ *is an algebraic space and a group scheme.*

Proof. Since X is quasi-projective, consider some compactification $\bar{X}$ of X and $\mathrm{Hilb}_{\bar{X}\times\bar{X}}$. □

Let $\mathrm{Aut^0(X)}$ denote the connected component of $\mathrm{Aut(X)}$ which contains an identity of the group.

Lemma 5. *Let X be a projective variety. There exists an inductive system of monomorphisms* $\mathrm{Aut^0(X_i)} \to \mathrm{Aut^0(X_{i+1})}$ *such that $X_0 = X$,* $\mathrm{Aut^0(X_i)}$ *acts regularly on some quasi-projective variety X_{i+1} which is a quasi-projective variety $X_i \setminus H$ where H is a hypersurface of X_i. The inductive limit of the system* $(\mathrm{Aut^0(X_i)})_{\mathrm{i\in I}}$ *is locally compact Lie group and an ind-algebraic space. Hence it is a pro-Lie group. If X is of Kodaira dimension ≥ 0, the birational automorphism group is locally algebraic.*

Proof. By Weil's theorem and the lemma above, we can construct an inductive system of monomorphisms $\mathrm{Aut^0(X_i)} \to \mathrm{Aut^0(X_{i+1})}$ such that $X_0 = X$, $\mathrm{Aut^0(X_i)}$ acts regularly on some quasi-projective variety X_{i+1} which is a quasi-projective variety $X_i \setminus H$ where H is a hypersurface of X_i. Since any $\mathrm{Aut^0(X_i)}$ is an algebraic group, the inductive limit is a Baire space in the complex topology, i.e., an inner point in the limit space is also an inner point some $\mathrm{Aut^0(X_i)}$. Thus the inductive limit of the system $(\mathrm{Aut^0(X_i)})_{\mathrm{i\in I}}$ is locally compact Lie group and an ind-algebraic space, which is also a pro-Lie group. When $\kappa(X) \geq 0$, there exists a maximal algebraic group birationally acting on X. Hence the inductive limit is an algebaic group which turns out to be an abelian group by Matsumura. □

Lemma 6. *Let X/S be a fibre space with the generic general fibe of Kodaira dimension ≥ 0. If X/S is neutral, then X/S is isotrivial.*

Theorem 5. *Let V be a quartic uniruled threefold. Then any deformation of a quartic threefold V over a curve with maximal variation contains only discrete quartic threefolds.*

9. Iitaka-Viehweg conjecture

Theorem 6. *Let X/S be a fibre space with the generic general fibre $X_{\bar{\eta}}$ of Kodaira dimension ≥ 0. For some $m > 0$,*

$$\kappa(\det\ f_*\omega_{X/S}^{\otimes m}) \geq \mathrm{var(X/S)}$$

Lemma 7. *Let X/S be a fibre space with the generic general fibre $X_{\bar{\eta}}$ of Kodaira dimension ≥ 0. There exists a fibre space Y/S such that*

(1) *Y is a cover over X,*
(2) *the generic general fibre $Y_{\bar{\eta}}$ of Y/S is of general type,*
(3) $\kappa(\det\ f_*\omega_{X/S}^{\otimes m}) = \kappa(\det\ g_*\omega_{Y/S}^{\otimes m})$.

Proof. Embed X/S into the trivial fibre space $S\times\mathbf{P}$. Let $i : X \to S\times\mathbf{P}$ be the embedding over S. Choose a general hyperplane H in $S\times\mathbf{P}$ such that the intersection $X \cap H$ is a non singular variety and $H = H_0 \times S$ is horizontal in $S \times \mathbf{P}$. Take a branch cover Y of X along H. Choose a hyperplane H such that Y/S has a general fibre of general type. Since $t_*\omega_{Y/S} = \omega_{X/S}(H)$, we have $(i \circ t)_*\omega_{Y/S}^{\otimes m} = i_*\omega_{X/S}^{\otimes} \otimes \mathcal{O}(H_0)$. Thus $\kappa(\det\ f_*\omega_{X/S}^{\otimes m}) = \kappa(\det\ g_*\omega_{Y/S}^{\otimes m})$ for $m > 0$. □

By Kollar's theorem, $\kappa(\det\ f_*\omega_{X/S}^{\otimes m}) = \kappa(\det\ g_*\omega_{Y/S}^{\otimes m}) \geq var(Y/S)$ for some $m > 0$.

Lemma 8. *Let Y/S and X/S be fire spaces. Assume that there exists an S-dominant rational map. Then* var(Y/S) $\geq$ var(X/S).

Proof. It suffices to prove that if var(Y/S) $= 0$, then var(X/S) $= 0$. Thus the extension $R(X)/R(S)$ is neutral, i.e., the epimorphism of the absolute Galois groups $\mathrm{Gal}(\bar{\mathrm{R}(\mathrm{X})}/\mathrm{R(X)}) \to \mathrm{Gal}(\bar{\mathrm{R}(\mathrm{S})}/\mathrm{R(S)})$ has a section. Since the generic general fibre of X/S is of Kodaira dimension ≥ 0 , X/S is isotrivial by Galois theory. □

Assume $\kappa(\omega_{X_{\bar{\eta}}}) \geq 0$. The theorem above implies $\kappa(\det\ f_*\omega_{X/S}^{\otimes m}) \geq$ var(X/S) for some $m > 0$. Hence there exists a canonical homomorphism $f^*(\det\ f_*\omega_{X/S}^{\otimes m})^{\otimes \ell} \hookrightarrow f^*(f_{ast}\omega_{X/S}^{\otimes m})^{\otimes r\ell} \to f^*f_*\omega^{\otimes rm\ell} \to \omega_{X/S}^{\otimes rm\ell}$, whose composition is not a zero map for smoe $\ell > 0$.

$$\begin{array}{ccc} X & \longrightarrow & V \\ \downarrow & & \downarrow \\ S & \longrightarrow & W \end{array}$$

Let V and W be the images of rational maps defined by $H^0(X, \omega_{X/S}^{\otimes rm\ell})$ and $H^0(S, (\det f_*\omega_{X/S}^{\otimes m})^{\otimes \ell})$. Looking at this commutative diagram of varieties, we have the following inequality: The dimension $\kappa(X_{\bar\eta})$ of the image of the generic general fibre $X_{\bar\eta}$ of X/S is less than that of the generic general fibre of V/W. Hence $\kappa(\omega_{X/S}) = \dim V \geq \kappa(X_{\bar\eta}) + \dim W$ and $\dim W = \kappa(\det f_*\omega_{X/S}^{\otimes m}) \geq \mathrm{var}(X/S)$.

Corollary 1.

$$\kappa(\omega_{X/S}) \geq \kappa(\omega_{X_{\bar\eta}}) + \mathrm{var}(X/S)$$

The original Iitaka conjecture is obtained by the result $\kappa(\omega_{X/S}) \geq 0$. We make use of the commutative diagram above. If $\kappa(S) \geq 0$ and $\kappa(X_{\bar\eta}) \geq 0$, then $\kappa(X) \geq 0$. Let V and W be the images of rational maps defined by $H^0(X, \omega_X^{\otimes m})$ and $H^0(S, \omega_S^{\otimes m})$, respectively.

Corollary 2. *Let X/S be a fibre space with $\kappa(X_{\bar\eta}) \geq 0$.*

$$\kappa(X) \geq \kappa(X_{\bar\eta}) + \kappa(S)$$

9.1. **Log-geometry.** Every things are to be defined over the complex number field. Let (X, D) be a pair of a projective non singular variety X and a divisor D with normal crossing only on X. The category of proper birational geometry of such objects is equivalent to that of the objects of semi-local rings with ring homomorphisms as arrows.

Let $(X, D_X)/(S, D_S)$ be a fibre space with all reduced fibres. It does not lose generality. Let $j : X \to \mathbf{P} \times S$ be a proper birational morphism over S. Let H be a horizontal very ample hyperplane on $\mathbf{P} \times S$. Take a ramified cover $Y = \operatorname{Spec} \mathcal{O}_X[T]/(T^d - a)$ along j^*H, where a is a non singular section of j^*H. We can choose H such that the induced log-variety (Y, D_Y) over S has the generic general fibre of maximal Kodaira dimension. Then we have $\kappa(\det f_*\omega_{(X,D_X)/(S,D_S)}^{\otimes m}) = \kappa(\det g_*\omega_{(Y,D_Y/(S,D_S))}^{\otimes m})$. By an analogue of Kollar theorem, $\kappa(\det g_*\omega_{(Y,D_Y/(S,D_S))}^{\otimes m}) \geq \mathrm{var}(Y, D_Y/(S, D_S))$. We apply $\mathrm{var}(Y, D_Y/(S, D_S) \geq \mathrm{var}(X, D_X/(S, D_S)$ to it. Let $1 \to G \to E \to K \to 1$ be an extension of a groupoid K by a groupoid G associated with that of semi-local rings which correspond to the fibre space $(X, D_X) \to (S, D_S)$. To an exact sequence $1 \to \mathrm{InnG} \to \mathrm{Aut}G \to \mathrm{Out}G \to 1$, we have an exact sequence

$$\mathrm{H}^1(K, \mathrm{Inn}G) \to \mathrm{H}^1(K, \mathrm{Aut}G) \to \mathrm{H}^1(K, \mathrm{Out}G),$$

i.e.,

$$\mathrm{Hom}(K, \mathrm{Inn}G) \to \mathrm{Hom}(K, \mathrm{Aut}G) \to \mathrm{Hom}(K, \mathrm{Out}G).$$

A groupoid extension is an element of $\mathrm{H}^1(K, G \to \mathrm{Aut}G)$, where $G \to \mathrm{Aut}G$ is a crossed module. We have

$$1 \to \mathrm{H}^2(K, \mathrm{Z}(G)) \to \mathrm{H}^1(K, G \to \mathrm{Aut}G) \to \mathrm{H}^1(K, \mathrm{Out}G).$$

Applying a log-version of Matsumura's result, the canonical representation $\rho : K \to \mathrm{Out}G$ turns out to be trivial after replacing a profinite groupoid associated to a finite morphism $(S', D_{S'} \to (S, D_S)$ in the following lemma. We can assume that the extension is neutral. Since we have $\mathrm{H}^2(K, \mathrm{Z}(G)) \to \mathrm{H}^1(K, G \to \mathrm{Aut}(G))$, the extension $1 \to G \to E \to K \to 1$ is given by pushing out a central extensio $1 \to \mathrm{Z}(G) \to E' \to K \to 1$. Hence E' is a semi-direct product $\mathrm{Z}(G) \times |K$, which is contained in a semi-direct product $G \times |K$. Thus this central extension is trivial. Therefore by pushing out this central extension, the extension $1 \to G \to E \to K \to 1$ is trivial. Hence we get $\mathrm{var}(Y, D_Y/(S, D_S) \geq \mathrm{var}(X, D_X/(S, D_S)$. Thus we have

$$\kappa(\det\ f_*\omega^{\otimes m}_{(X,D_X)/(S,D_S)}) = \kappa(\det\ g_*\omega^{\otimes m}_{(Y,D_Y/(S,D_S)}) \geq$$
$$\mathrm{var}(Y, D_Y/(S, D_S) \geq \mathrm{var}(X, D_X/(S, D_S).$$

Theorem 7.

$$\kappa(\det\ f_*\omega^{\otimes m}_{(X,D_X)/(S,D_S)}) \geq \mathrm{var}((X, D_X)/(S, D_S)).$$

Corollary 3. *If a canonical divisor K_X on X is numerically $\mathbf{Q}$-equivalent to an effective $\mathbf{Q}$-divisor on X, then $\kappa(\omega_X) \geq 0$.*

Proof. Take the Albanese map $\alpha : X \to \mathrm{Alb}(X)$. When $q > 0$, $\kappa(\omega_{X/A}) \geq \kappa(\omega_{X_{\bar\eta}}) + \mathrm{var}(X/A)$ implies $\kappa(\omega_X) \geq 0$, since A which is the image of α is of Kodaira dimension ≥ 0 and $\kappa(\omega_{X_{\bar\eta}}) \geq 0$ by induction. When $q = 0$, numerical equivalence is linear equivalence. □

9.2. **Minimal Models.** We describe a strategies for minimal models. We work in the category such that the objects are log-varieties (X, D) and the morphisms are ordinary morphisms $f : (X, D) \to (Y, E)$ such that $f^* : K_Y + E \to K_X + D$ in $\mathrm{Pic}X \otimes \mathbf{Q}$. Let X_1 be a projective variety over the field of complex numbers and D_1 a Q-divisor on X_1 with the support normally crossing. By classical geometry and Hironaka's resolution, we have a fibre space $f : X \to S$ with all the fibres curves, where X and S are non singular projective varieties and X is a modification of X_1. Furthermore, (X, D) is log-geometrically equivalent to (X_1, D_1). Since the fibre of X/S is a curve, Iitaka-Vieweg conjecture $\kappa(\det f_*\omega^{\otimes m}_{(X,D)/S}) \geq \mathrm{var}(X, D)/S$ holds from Hodge theory. It implies that $\kappa(\omega_{(X,D)/S}) \geq \kappa((X, D)_{\bar{\eta}}) + \mathrm{var}(X, D)/S$. If, moreover, $\kappa((X, D)) \geq 0$, then $K_X + D$ has an expression in the form $f^*(K_S + G)$ plus something effective in $\mathrm{Pic}X \otimes \mathbf{Q}$, where G is a $\mathbf{Q}$-divisor on S. By induction assumption, assume that there exists a normal projective variety (T, H) with log-terminal singularities only and $\mathbf{Q}$-divisor H on T such that

(1) (T, H) is equivalent to (S, D) in log-geometry(i.e. log-modification),
(2) $K_T + H$ is nef (resp. abundant),

Note that the family of curves over a dense open set of T is induced over T through X/S. Taking a finite extension and its Galois extension, we have a normal projective variety U with log-terminal singularities only and a family Y of nordal curves over U. We get a log-variety (Z, D_Z) by an automorphism of a Galois group $\mathrm{A}ut(U/T)$ making use of de Jong and Abramovich methods. Then $g : Z \to T$ is a fibre space with all the fibres nodal curves. It is well-known (resp. conjectured) that $g_*\omega^{\otimes m}_{Z/T}$ is nef (resp. abundant). It should be shown that $g_*\omega^{\otimes m}_{(Z,D_Z)/(T,H)}$ is nef (resp. abundant). There is a natural surjective homomorphism $g^*g_*\omega^{\otimes m}_{(Z,D_Z)/(T,H)} \to \omega_{(Z,D_Z)/(T,H)}$. Then since $K_T + H$ is nef (resp. abundant), induction completes, hence we get a nef (resp. abundant) model (Z, D_Z) for (X_1, D_1). It is enough to show $g_*\omega^{\otimes m}_{(Z,D_Z)/(T,H)}$ is abundant. We may choose maximal $\omega_{(T,H)}$ such that $\kappa(\omega_{(Z,D_Z)/(T,H)}) \geq 0$. Take a cover (W, D_W) of (Z, D_Z) along an ample hyperplane horizontal with respect to Z/T and take Kawamata cover T_1 of T along an ample hyperplane. We get the pull-back (V, D_V) of (W, D_W) through T_1/T. Then if $\kappa(T_1) = \dim T_1$, $\kappa((V, D_V)) = \dim V$. Hence $\omega(V, D_V)$ is able to have a minimal model. Hence $\omega_{(V,D_V)}$ is nef and big, so abundant. Thus for the induced map h, $h_*\omega^{\otimes m}_{(V,D_V)}$ is abundant, so $h_*\omega^{\otimes m}_{(V,D_V)/(T_1,D_{T_1})}$ is also abundant. Then $g_*\omega^{\otimes m}_{(Z,D_Z)/(T,H)}$ is a direct factor of $h_*\omega^{\otimes m}_{(V,D_V)/(T_1,D_{T_1})}$. Hence it is abundant.

Let (X, D) a log-variety. Assume $K_X + D$ is weakly positive. For simplicity sake, assume there exists a fibre space $(X, D) \to S$ with all the fibre of nodal curves. In the same argument above, $\kappa(\omega_{(X,D)/S}) \geq 0$. Since $K_X + D$ is weakly positive, there exists $\mathbf{Q}$-divisor D_S on S such that $K_S + D_S$ is weakly positive. By inductin, $\kappa(K_S + D_S) \geq 0$ and $\kappa((K_X + D)|_{\bar{\eta}}) \geq 0$. We obtain $\kappa(\omega_{(X,D)/(S,D_S)}) \geq \kappa((K_X + D)|_{\bar{\eta}}) \geq 0$. Thus $\kappa(K_X + D) \geq 0$ if $K_X + D$ is weakly positive.

Proposition 1. *If $K_X + D$ is weakly positive, then $\kappa(K_X + D) \geq 0$. In particular, if K_X is weakly positive (i.e. pseudo effective), then $\kappa(K_X) \geq 0$.*

Let (X, D) be a log-variety. Assume (X, D) is non log-uniruled. If $K_X + D$ is weakly positive, then $\kappa(K_X + D) \geq 0$. If $K_X + D$ is not weakly positive, put $\epsilon_0 = \inf\{t | K_X + D + tH \text{is weakly positive}, t > 0\}$. Here H is an ample divisor on X. We have $\epsilon_0 > 0$, $\epsilon_0 \in \mathbf{Q}$ and $K_X + D + \epsilon_0 H$ is weakly positive. Hence $\kappa(K_X + D + \epsilon_0 H) \geq 0$. Thus we have a rational map defined by $m(K_X + D + \epsilon_0 H)$ for $m >> 0$. Hence (X, D) is log-uniruled.

Proposition 2. *Let (X, D) be a log-variety with $\kappa(K_X + D) = -\infty$. (X, D) is a log-uniruled variety. In particular, if X is a variety with $\kappa(X) = -\infty$, then X is uniruled.*

Let X be a projective non singular variety with no global sections of any multiple of Ω_X, i.e., $H^0(X, \Omega_X^{\otimes m}) = 0$ for all $m > 0$. Then $\kappa(\omega_X) = -\infty$. Hence K_X is not weakly positive and so X is a uniruled variety. X has a canonical structure with maximally rationally connected fibres, i.e., MRC-fibration. If X is not a rationally connected variety, there exists a surjective morphism f from X onto a non uniruled variety S. By induction, K_S is weakly positive and $\kappa(\omega_S) \geq 0$. Hence $H^0(S, \Omega_S^{\otimes m}) \neq 0$ and we have an injection $H^0(S, \Omega_S^{\otimes m}) \hookrightarrow H^0(X, \Omega_X^{\otimes m})$, which is a contradiction.

Proposition 3. *Let X be a projective non singular variety with $H^0(X, \Omega_X^{\otimes m}) = 0$ for all $m > 0$. Then X is a rationally connected variety.*

9.3. **Curves of a fixed genus.** Let (X, D) be a log-variety of $\kappa(\omega_{(X,D)}) = \dim X$. Take a fibre space $(X, D)/(S, D_S)$ of relative dimension 1 such that $\kappa(\omega_{(S,D_S)}) = \dim S$. By induction, we shall show the upper boundedness of the intersection numbers between the canonical divisor and curves ofa given genus. Define a curve C of log-genus g in (X, D) to be $\deg_C(\Omega_C < D >) = 2g - 2$. Then the log-genus of the image of C in S is at most g. By induction, the intersection number $(\omega_{(S,D_S)}, C_S)$ is bounded above except for curves included in a hypersurface in S. By Iitaka's formula, $\kappa(X_{\bar{\eta}}) + \dim S \geq \kappa((X, D)) = \dim X$. Hence the general fibre of $(X, D)/(S, D_S)$ is of Kodaira dimension 1. For a general curve C_S, $\kappa((X, D)_{C_S}) \geq \kappa((X, D)) - \dim S + 1$. Thus $\kappa((X, D)_{C_S}) = 2$. By Miyaoka-Sakai inequality, the intersection number of $(\omega_{(X,S)}, C)$ is bounded above since surfaces over the other image curve have the same topological invariants, say, $c_2((X, D)_{C_S})$, $c_1((X, D)_{C_S})$ except for neglectible curves.

Proposition 4. *Let (X, D) be a log-variety of $\kappa((X, D)) = \dim X$. Given a fixed log-genus g, there exists an upper bound for the intersection number $(C, K_X + D)$ which is independent of curves C on (X, D) of log-genus g.*

Problem 1. *Let S/B be an elliptic fibre space and B the projective space $\mathbf{P}^1$. Let $\kappa(S) \geq 0$. Given a number g, construct a family of k_n-folds curves C_n on S over B with ordinary genus of g such that k_n tends to infinity.*

10. Group Theoretic Geometry

In this section we shall propose a construction of a new geometry called group theoretic geometry. Possibly it might be a variation of a representation theory of groups. We grasp the concept of a variety in this sense as follows: Given a field F, we patch subrings (R_i) of the field F with the same total quotient field $Q(R_i) = F$ in such a way that there exists a topological space $X = \cup_i U_i$ such that $U_i = \mathrm{Spec} R_i$. Here the subrings (R_i) form an inductive system and a generic point is represented by $F = \lim\limits_{\rightarrow} R_i$. For an affine variety R_i and an irreducible divisor $V(f)$, the generic point of $V(f)$ is defined by the total quotient of $R_i/(f)$.

Let Γ be a profinite group and $\{H_i\}$ a family of profinite normal closed subgroups of Γ. We patch quotient groups Γ/H_i such that there exists a topological space $X = \cup_i U_i$, where $\pi_1(U_i) = \Gamma/H_i$. The Γ/H_i form a projective system and the limit is $\Gamma = \lim\limits_{\leftarrow} \Gamma/H_i$. For an affine variety $U_i = \mathrm{Spec} R_i$ and a divisor D of U_i, $\pi_1(U_i)$ is a factor group of a profinite group Γ_D of a generic poit of D. We have $\Gamma_D = \lim\limits_{\leftarrow\, g \neq f} \pi_1(R_i[f^{-1}]/(g))$, where $V(g) = D$.

10.1. Affine scheme.

Lemma 9. *Let A_p be a local ring and p a prime ideal. Let $\hat{A}_p$ be a completion of A_p. Then $\hat{A}_p \to A_p$ is faithfully flat and etale. Hence the homomorphism $\pi_1(\mathrm{Spec}\ \hat{A}_p) \to \pi_1(\mathrm{Spec}\ A_p)$ is injective.*

Proof. Since $\hat{A}_p/p \cong A_p/p$, Spec $\hat{A}_p \to$ Spec A_p is unramified and faithfully flat. Hence it is an etale covering. □

Theorem 8. *([?]) Let $Y =$ Spec A, A a complete noetherian local normal ring with k a residue field, X a proper Y-scheme, $X_0 = A \otimes_A k$, a_0 a geometric point and a the corresponding point of X. Then the canonical homomorphism $\pi_1(X_0, a_0) \to \pi_1(X, a)$ is an isomorphism.*

Corollary 4. *Since $\hat{A}_p$ is a complete noetherian local ring, one has* $\pi_1(\text{Spec } A_p/p) \cong \pi_1(\text{Spec } \hat{A}_p)$

Proof. $\hat{A}_p/p \cong A_p/p$. □

Let k be a field and A a k-algebra of finite type. Let $X = \text{Spec } A$. Assume $A \subset Q(A)$.

Let Γ be the absolute Galois group $\pi_1(Q(A), \overline{Q(A)})$. Let $f \in A$. Then $\pi_1(A[f^{-1}])$ is a quotient Γ/N_f of Γ.

Proposition 5. *Let S be an irreducible locally noetherian normal scheme with K the function field. Let Ω be an algebraically closed extension of K. Let a' be a geometric point of* Spec K *and a a geometric point of S. Then the homomorphism*

$$\pi_1(\text{Spec } K, a') \longrightarrow \pi_1(S, a)$$

is surjective. The kernel corresponds to the subextension $\bar{K}/K$ composed by finite extensions in Ω unramified over S.

Proof. The maximal etale covering over S induces an unramified extension of K. Let N be the corresponding normal subgroup. One has an isomorphism $\pi_1(S, a) \cong \Gamma/N$. □

Corollary 5. *Let $U \subset X$ be an open immersion. Then the homomorphism $\pi_1(U) \longrightarrow \pi_1(X)$ is surjective.*

Proposition 6. *Let $f : Y \to X$ be an etale covering and b, a geometric points which corresponds by f over Y , X. Then the homomorphism $\pi_1(Y, b) \to \pi_1(X, a)$ is injective.*

Definition 4. *([?]) A morphism $f : X \to S$ is said to be separable if every fibre is reduced.*

Proposition 7. *Let $f : X \to S$ be a proper separable morphism between locally noetherian schemes. Then* Spec $f_*\mathcal{O}_X \to S$ *is an etale covering.*

Theorem 9. *([?]) Let $f : X \to Y$ be a proper separable morphism with Y locally noetherian connected scheme. Suppose $f_*\mathcal{O}_X = \mathcal{O}_Y$. Let X' be an etale covering of X. There exists an etale covering Y' such that $X' \cong X \times_Y Y'$, if and only if $\overline{X'_y}$ admits a section over $\overline{X_y}$.*

Corollary 6. *([?]) Let $\bar{a}$ be a geometric poit of $\overline{X_y}$, a its image in X and b its image in Y. Then the the following sequence is exact:*

$$\pi_1(\overline{X_y}, \bar{a}) \to \pi_1(X, a) \to \pi_1(Y, b) \to 1$$

Corollary 7. *([?]) Let k be an algebraically closed field, X and Y connected schemes over k. Suppose that X is proper over k and Y is locally noetherian. Let a be a geometric point of X, b a geometric point of Y with values in the same algebraically closed extension K of k. Let $c = (a, b)$. The homomorphism $\pi_1(X \times_k Y, c) \to \pi_1(X, a) \times \pi_1(Y, b)$ is induced by the homomorphisms between the fundamental groups associated to projections $X \times_k Y \to X$ and $X \times_k Y \to Y$. This homomorphism is an isomorphism.*

Proposition 8. *([?]) Let $f : X \to Y$ be a proper separable morphism with Y locally noetherian connected scheme. Suppose $f_*\mathcal{O}_X = \mathcal{O}_Y$. The homomorphism $\pi_1(X) \to \pi_1(\text{Spec } f_*\mathcal{O}_X)$ is surjective and $\pi_1(\text{Spec } f_*\mathcal{O}_X) \to \pi_1(Y)$ is injective.*

Let $X = \text{Spec } A$ be a locally noetherian normal irreducible affine scheme and Γ a profinite group $\pi_1(K)$. Let $\pi_1(A[f^{-1}]) = \Gamma/N_f$, where N_f is a normal closed subgroup in Γ. One has $\pi_1(A_p) = \pi_1(\lim\limits_{\to f \notin p} A[f^{-1}]) = \lim\limits_{\leftarrow f \notin p} \pi_1(A[f^{-1}]) = \lim\limits_{\leftarrow f \notin p} \Gamma/N_f$. So $\pi_1(A_p) = \Gamma/\cap_{f \notin p} N_f$. Let N_p denote $\cap_{f \notin p} N_f$. Next one has

$$\pi_1(A_p/p) \hookrightarrow \Gamma/N_p.$$

Hence there exists a subgroup Γ_p of Γ such that $\pi_1(A_p/p) \cong \Gamma_p/N_p$. Let $\Gamma(p)$ denote Γ_p/N_p. Let $p \subset q$ be prime ideals. Since $A_q \subset A_p$, $\pi_1(A_p) \to \pi_1(A_q)$ is surjective. Hence $N_p \subset N_q$. Characterizing N_p, one defines a set $X = \{N_p\}$. One gives the closure of a point N_p, $V(N_p) = \{N_q | N_p \subset N_q\}$. Let M be a normal closed subgroup of Γ such that $N \subset M$. Let $V(M) = \{N_p | M \subset N_p\}$. One has $\Gamma/M \cong \pi_1(\lim\limits_{\to f \in S} A_f)$. Let a be an ideal of A. For $V(a)$ take $\Delta = \cap\{\Gamma_p | \Gamma_p \text{ associate to } p \in V(a)\}$. One has $\Delta/\Delta \cap N \cong \pi_1(A/a)$.

10.2. **Proper Smooth morphisms.**

Theorem 10. *([?]) Let $f : X \to Y$ be a proper smooth morphism. Suppose $f_*\mathcal{O}_X = \mathcal{O}_Y$ and Y is locally noetherian. Assume that y_0 and y_1 are points of Y such that $y_0 \in \overline{y_1}$ and that $\overline{X_0}$ and $\overline{X_1}$ are geometric fibres. Then the specialization homomorphism $\pi_1(\overline{X_1}) \to \pi_1(\overline{X_0})$ is an isomorphism if $k(y_0)$ is of characteristic 0 (resp. $\pi_1(\overline{X_1})^{(p)} \to \pi_1(\overline{X_0})^{(p)}$ if $k(y_0)$ is of characteristic p).*

Proposition 9. *Let $f : X \to S$ be a proper smooth surjective morphism between complex varieties. Suppose $f_*\mathcal{O}_X \cong \mathcal{O}_Y$. Let $g : T \to S$ be a morphism of finite type. Then $\pi_1(X \times_S T) \cong \pi_1(X) \times_{\pi_1(S)} \pi_1(T)$.*

Proof. By base change $f_T : X_T \to T$ is a proper smooth morphism. Hence $\pi_1(X \times_S T) \cong \pi_1(X) \times_{\pi_1(S)} \pi_1(T)$. $\square$

Theorem 11. *Let $f : X \to S$ be a proper smooth surjective morphism between complex non singular varieties. Suppose that $f_*\mathcal{O}_X \cong \mathcal{O}_Y$ and that $\pi_1(S, a) = 1$. Then $f : X \to S$ is trivial.*

Proof. One denotes the rational function field of S by $R(S)$. Put $T = \mathrm{Spec}R(S)$. Since $f_T : X_T \to T$ is a proper smooth morphism, $\pi_1(X_T) \cong \pi_1(X) \times \pi_1(T)$. We will show a proof in the following several steps. $\square$

Lemma 10. *Let X be a non singular variety and U, V open varieties. Assume* $\mathrm{codim}(U \cup V, X) \geq 2$. *Then the following commutative square is cartesian:*

$$\begin{array}{ccc} \pi_1(U \cap V) & \to & \pi_1(U) \\ \downarrow & & \downarrow \\ \pi_1(V) & \to & \pi_1(X) \end{array}$$

Proof. Let Γ be the absolute Galois group of the function field $R(X)$ of X and N_X, N_U, N_V normal closed subgroups of Γ which correspond to varieties X, U, V, respectively. Since $N_U/N_U \cap N_V \cong N_U N_V/N_V$, one has the map of the following exact sequences the lower square of which is cartesian:

$$\begin{array}{ccc} 1 & & 1 \\ \downarrow & & \downarrow \\ N_U/N_U \cap N_V & \cong & N_U N_V/N_V \\ \downarrow & & \downarrow \\ \Gamma/N_U \cap N_V & \to & \Gamma/N_V \\ \downarrow & & \downarrow \\ \Gamma/N_U & \to & \Gamma/N_U N_V \\ \downarrow & & \downarrow \\ 1 & & 1 \end{array}$$

$\square$

Corollary 8. *Let $f : X \to S$ be a surjective separable connected morphism between non singular varieties. Let U be an open subvariety of X. Assume that $f(U) = S$. Let η be the generic point of S and X_η, U_η generic fibres of X/S, U/S, respectively. Then the following commutative square is cartesian:*

$$\begin{array}{ccc} \pi_1(U_\eta) & \to & \pi_1(U) \\ \downarrow & & \downarrow \\ \pi_1(X_\eta) & \to & \pi_1(X) \end{array}$$

Proof. Let Γ be the absolute Galois group of the function field $R(X)$ of X and N_X, N_U, N_{X_η} normal closed subgroups of Γ which correspond to varieties X, U, X_η, respectively. Let W be an open subvariety of S. One has $N_U/N_U \cap N_{X_W} \cong N_U N_{X_W}/N_{X_W}$. Since $\lim_{\leftarrow} W = \eta$, one has $\lim_{\overleftarrow{W}} N_U/N_U \cap N_{X_W} \cong \lim_{\overleftarrow{W}} N_U N_{X_W}/N_{X_W}$. Since $N_U/N_U \cap N_{X_\eta} \cong N_U N_{X_\eta}/N_{X_\eta}$, one has the map of the following exact sequences the lower square of which is cartesian:

$$\begin{array}{ccc} 1 & & 1 \\ \downarrow & & \downarrow \\ N_U/N_{U_\eta} & \cong & N_X/N_{X_\eta} \\ \downarrow & & \downarrow \\ \Gamma/N_{U_\eta} & \to & \Gamma/N_{X_\eta} \\ \downarrow & & \downarrow \\ \Gamma/N_U & \to & \Gamma/N_X \\ \downarrow & & \downarrow \\ 1 & & 1 \end{array}$$

□

Lemma 11. *Let $f : X \to S$ be a proper smooth connected surjective morphism between varieties. Then*

$$\begin{array}{ccc} 1 & & 1 \\ \downarrow & & \downarrow \\ \pi_1(\overline{U}_\eta) & \cong & \pi_1(\overline{X}_\eta) \\ \downarrow & & \downarrow \\ \pi_1(U_\eta) & \to & \pi_1(X) \\ \downarrow & & \downarrow \\ \pi_1(\eta) & \to & \pi_1(S) \\ \downarrow & & \downarrow \\ 1 & & 1 \end{array}$$

Proof. One has the following commutative diagram:

$$\begin{array}{ccc} 1 & & 1 \\ \downarrow & & \downarrow \\ \pi_1(\overline{X}_\eta) & \cong & \pi_1(\overline{X}_\eta) \\ \downarrow & & \downarrow \\ \pi_1(X_\eta) & \to & \pi_1(X) \\ \downarrow & & \downarrow \\ \pi_1(\eta) & \to & \pi_1(S) \\ \downarrow & & \downarrow \\ 1 & & 1 \end{array}$$

Thus one interprets the commutative diagram above in the following one:

$$\begin{array}{ccc} 1 & & 1 \\ \downarrow & & \downarrow \\ M/N_X & \cong & N/N_{X_\eta} \\ \downarrow & & \downarrow \\ \Gamma/N_X & \to & \Gamma/N_{X_\eta} \\ \downarrow & & \downarrow \\ \Gamma/M & \to & \Gamma/N \\ \downarrow & & \downarrow \\ 1 & & 1 \end{array}$$

Hence one has $M/N_X \cong N/N_{X_\eta}$.

This implies $M/N_U \cong N/N_{U_\eta}$.

Consider the following commutative diagram:

$$\begin{array}{ccc} 1 & & 1 \\ \downarrow & & \downarrow \\ N_{X_\eta}/N_{U_\eta} & \cong & N_X/N_U \\ \downarrow & & \downarrow \\ N/N_{U_\eta} & \to & M/N_U \\ \downarrow & & \downarrow \\ N/N_{X_\eta} & \cong & M/N_X \\ \downarrow & & \downarrow \\ 1 & & 1 \end{array}$$

Since the lower and upper horizontal arrows are isomorphic, the midle horizontal arrow is an isomorphism by the precedent lemma. □

Lemma 12. *Let $g : T \to S$ be an arbitrary morphism and U an arbitrary open subvariety of X. Assume $\pi_1(U_T) = \pi_1(\overline{U}_\eta) \times \pi_1(T)$. Then there exists a variety F such that $X \cong F \times S$ as S-scheme.*

Hence we complete a proof of the theorem above.

We use the tools of L.Breen([?],[?],[?]) in the proof of the following theorem.

Theorem 12. *Let $f : X \to S$ be a proper smooth surjective morphism between non singular complex varieties. Assume*

: *(a) $f_*\mathcal{O}_X = \mathcal{O}_S$*

: *(b) the generic general fibre of $f : X \to S$ is of general type.*

: *(c) $S = S_1 \times S_2$, where S_1, S_2 are varieties.*

: *(d) for any point $b \in S_2$ the restriction $f_{1b} : X_{1b} \to S_1 \times \{b\}$ of $f : X \to S$ has no fixed part over S_1.*

Then there exists S_3 such that for any point $a \in S_1$ the base change $f_{2a,S_3} : X_{2a,S_3} \to \{a\} \times S_3$ of the restriction of $f_{2a} : X_{2a} \to S_2 \times \{a\}$ is trivial.

Proof. We will give an outline of a proof in several steps in the following lemmas and propositions. □

Proposition 10. *Let* $1 \to G \to E \to P \to 1$ *be an extension of profinite groups. There exist the following commutative diagram and exact sequence:*

$$\begin{array}{ccccccccc} 1 & \to & G & \to & E & \to & P & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & \mathrm{Inn}G & \to & \mathrm{Aut}G & \to & \mathrm{Out}G & \to & 1 \end{array}$$

and

$$0 \to H^1(P, \mathrm{Z}G) \to H^0(P, (G \to \mathrm{Aut}G)) \to H^0(P, \mathrm{Out}G) \to$$

$$H^2(P, \mathrm{Z}G) \to H^1(P, (G \to \mathrm{Aut}G)) \to H^1(P, \mathrm{Out}G)$$

Lemma 13. *Let* $1 \to G \to E \to P \to 1$ *be an extension of profinite groups. Let* $Q \triangleleft P$ *be a normal closed group of* G. *The following sequence is exact:*

$$1 \to H^1(P/Q, \mathrm{Z}G^Q) \to H^1(P, \mathrm{Z}G) \to H^1(Q, \mathrm{Z}G)^{P/Q}$$

Proof. It is the Edge sequence of Serre Spectral sequence([Se]). □

In particular, one has the following lemma.

Lemma 14. *Let* $P = P_1 \times P_2$. *The following sequence is exact: for* $\{i, j\} = \{1, 2\}$

$$0 \to H^1(P_j, \mathrm{Z}G^{P_i}) \to H^1(P, \mathrm{Z}G) \to H^1(P_i, \mathrm{Z}G)^{P_j}$$

Lemma 15. *For* $\xi \in H^1(P, (G \to \mathrm{Aut}G))$, *let* ξ_i *be the image element of the following composite homomorphism*

$$H^1(P, (G \to \mathrm{Aut}G)) \overset{res}{\to} H^1(P_i, (G \to \mathrm{Aut}G)) \to H^1(P_i, \mathrm{Out}G)$$

Then there exists a profinite subgroup P_i^* *of* P_i *such that* ξ_i *is a distinguished element of* $H^1(P_i^*, \mathrm{Out}G)$. *Hence there exists* $\zeta_i \in H^2(P_i^*, \mathrm{Z}G)$ *such that the image of* ζ_i *is res*(ξ) *by* $H^2(P_i^*, \mathrm{Z}G) \to H^1(P_i^*, (G \to \mathrm{Aut}G))$.

Proof. Since $\mathrm{Out}G$ is a finite group, it is possible to put $P_i^* = \ker(\xi_i : P_i \to \mathrm{Out}G)$. □

Lemma 16. *One can take a profinite subgroup* P_i^{**} *of* P_i^* *such that* $H^1(P_i^{**}, \mathrm{Z}G) = 0$ *for* $i = 1.2$. *Denoting by* $P^{**} = P_1^{**} \times P_2^{**}$. *One has* $H^1(P^{**}, \mathrm{Z}G) = 0$.

Proof. Take $P_i^{**} = \bigcap \ker(P_i^* \to \mathrm{Z}G) \supset [P_i^*, P_i^*] \neq 1$. Then $Hom(P^{**}, \mathrm{Z}G) = \Pi_{i \in 1,2} Hom(P_i^{**}, \mathrm{Z}G) = 1$. □

Lemma 17. *([Se]) Let $Q \triangleleft P$ be a normal profinite subgroup of P. If $H^1(P, \mathrm{Z}G) = 0$, then there exists an exact sequence:*

$$0 \to H^2(P/Q, \mathrm{Z}G^Q) \to H^2(P, \mathrm{Z}G) \to H^2(Q, \mathrm{Z}G)^{P/Q}$$

Proof of theorem. Replace P^{**}, P_i^{**}, P_j^{**} by P, P_i, P_j, respectively. Hence one can assume $H^1(P, \mathrm{Z}G) = 0$, $H^1(P_i, \mathrm{Z}G) = 0$, $H^1(P_j, \mathrm{Z}G) = 0$. Let $\xi \in H^1(P, (G \to \mathrm{Aut}G))$ be an extension :$1 \to G \to E \to P \to 1$ as in the theorem. The image of ξ by $H^1(P, (G \to \mathrm{Aut}G)) \to H^1(P, \mathrm{Out}G)$ is a distinguished element.

Hence there exists an element $\zeta \in \overline{H}^2(P, \mathrm{Z}G)$ such that $\alpha(\zeta) = \xi$ by $\overline{H}^2(P, \mathrm{Z}G) \overset{\alpha}{\to} \overline{H}^1(P, (G \to \mathrm{Aut}G))$. Here the overline of $\overline{H}(P, \mathrm{Z}G)$ indicates the reduced cohomology. Consider the following exact sequence:

$$0 \to \overline{H}^2(P_j, \mathrm{Z}G^{P_i}) \overset{\psi_j}{\to} \overline{H}^2(P, \mathrm{Z}G) \overset{\phi_i}{\to} \overline{H}^2(P_i, \mathrm{Z}G)^{P_j}$$

Take the extension $\xi \in H^1(P, (G \to \mathrm{Aut}G))$ of P such that $\phi_1(\xi)$ has no fixed part restricting to P_1. Suppose the image $\zeta - \psi_1\phi_1(\zeta)$ is not a distinguished element in the cohomology $\overline{H}^2(P, \mathrm{Z}G)$. Then the image $\zeta - \phi(\psi_1(\phi_1(\zeta)))$ is a distinguished element in the cohomology $\overline{H}^2(P_1, \mathrm{Z}G)^{P_2}$. Hence there exists an element η_2 such that $\psi_2(\eta_2) = \zeta - \psi_1\phi_1(\zeta)$. Consider the following exact sequence: $0 \to \overline{H}^2(P, \mathrm{Z}G) \to \overline{H}^1(P, (G \to \mathrm{Aut}G)) \to \overline{H}^1(P, \mathrm{Out}G)$. Thus the extension ξ has a non void fixed part restricting to P_1, which is a contradiction. □

Therefore we complete a proof of the theorem above.

11. Hodge Conjecture

In this section we show that the Hodge conjecture and a part of the Tate conjecture hold. Since it is difficult to find algebraic cycles in general, the strategy is to proceed by induction argument to vanish a certain subspace of a cohomology of an open affine subvariety of an affine variety which is obtained excluding a general hyperplane from a given variety.

In Case of Non Sigular Varieties. Let k be a field with an algebraic closure $\overline{k}$ and X a smooth geometrically irreducible variety over k.

There exists the canonical cycle map for $\ell \neq \mathrm{char}k$

$$cl_\ell^r : \mathrm{CH^r(X)} \longrightarrow \mathrm{H^{2r}_{et}(X_{\overline{k}}, \mathbf{Q}_\ell(r))}$$

This image is included in the fixed part
$\mathrm{H^{2r}_{et}(X_{\overline{k}}, \mathbf{Q}_\ell(r))^{G_k}}$

where $G_{\mathrm{k}} = \mathrm{Gal}(\overline{k}/k)$.

Tates conjecture says that if k is finitely generated as a field, the image of cl_ℓ^{r} generates $\mathrm{H}^{2\mathrm{r}}_{\mathrm{et}}(\mathrm{X}_{\overline{\mathrm{k}}}, \mathbf{Q}_\ell(\mathrm{r}))^{\mathrm{G_k}}$. Fix an isomorphism $\iota : \overline{\mathbf{Q}}_\ell \to \mathbf{C}$.

Let $k = \mathbf{C}$. One has the canonical cycle map

$$cl^r : \mathrm{CH}^{\mathrm{r}}(\mathrm{X}) \longrightarrow \mathrm{H}^{2\mathrm{r}}(\mathrm{X}(\mathbf{C}), \mathbf{Q}(2\pi \ \mathrm{i})^{\mathrm{r}})$$

This image is included into

$$\mathrm{H}^{2\mathrm{r}}(\mathrm{X}(\mathbf{C}), \mathbf{Q}(2\pi \ \mathrm{i})^{\mathrm{r}}) \cap \mathrm{H}^{\mathrm{r,r}}(\mathrm{X}(\mathbf{C}))$$

Hodge conjecture says that the image of cl^{r} generates $\mathrm{H}^{2\mathrm{r}}(\mathrm{X}(\mathbf{C}), \mathbf{Q}(2\pi \ \mathrm{i})^{\mathrm{r}}) \cap \mathrm{H}^{\mathrm{r,r}}(\mathrm{X}(\mathbf{C}))$.

Let U be a smooth quasiprojective variety over k.

The images of the canonical cycle maps are

$$\begin{cases} \mathrm{H}^{2\mathrm{r}}_{\mathrm{et}}(\mathrm{U}_{\overline{\mathrm{k}}}, \mathbf{Q}_\ell(\mathrm{r}))^{\mathrm{G_k}} & \text{for finitely generated } k \\ F^r\mathrm{H}^{2\mathrm{r}}(\mathrm{U}, \mathbf{C}) \cap \mathrm{W}_{2\mathrm{r}}\mathrm{H}^{2\mathrm{r}}(\mathrm{U}, \mathbf{Q}(\mathrm{r})) & \text{for } k = \mathbf{C} \end{cases}$$

Let U be a smooth quasi-projective variety over k and X a smooth projective compactification of U. One denotes by cl_* the following cycle maps cl_{DR}, cl_ℓ, cl_H;

: (a) $\mathbf{\Gamma}_{\mathrm{DR}}(\mathrm{H}^{2\mathrm{r}}_{\mathrm{DR}}(\mathrm{U})(\mathrm{r})) = \mathrm{W}_0(\mathrm{H}^{2\mathrm{r}}_{\mathrm{DR}}(\mathrm{U})(\mathrm{r})) \cap \ \mathrm{F}^0(\mathrm{H}^{2\mathrm{r}}_{\mathrm{DR}}(\mathrm{U})(\mathrm{r}))$

: (b) $\mathbf{\Gamma}_\ell(\mathrm{H}^{2\mathrm{r}}_\ell(\mathrm{U})(\mathrm{r})) = \mathrm{H}^{2\mathrm{r}}_\ell(\mathrm{U})(\mathrm{r})^{\mathrm{G_k}} \cap \ \mathrm{W}_0(\mathrm{H}^{2\mathrm{r}}_\ell(\mathrm{U})(\mathrm{r})))$

: (c) $\mathbf{\Gamma}_H(\mathrm{H}^{2\mathrm{r}}_\sigma(\mathrm{U})(\mathrm{r})) = \mathrm{W}_0(\mathrm{H}^{2\mathrm{r}}_\sigma(\mathrm{U})(\mathrm{r}) \cap \ \mathrm{F}^0(\mathrm{H}^{2\mathrm{r}}_\sigma(\mathrm{U})(\mathrm{r})) \otimes \mathbf{C})$

Lemma 18. *It suffices to prove it for a smooth affine variety over k.*

Proof. It is well known that it is enough to treat it in the case of $\dim X = 2d$.

Choose a smooth irreducible hyperplane Y on X. Thus $X - Y$ is a smooth affine variety.

One obtains the following commutative diagram:

$$\begin{array}{ccc} \mathrm{CH}^{\mathrm{d}-1}(\mathrm{Y}) & \longrightarrow & \mathbf{\Gamma}_*\mathrm{H}^{2\mathrm{d}}_{*,\mathrm{Y}}(\mathrm{X})(\mathrm{d}) \\ \downarrow & & \downarrow \\ \mathrm{CH}^{\mathrm{d}}(\mathrm{X}) & \longrightarrow & \mathbf{\Gamma}_*\mathrm{H}^{2\mathrm{d}}_*(\mathrm{X})(\mathrm{d}) \\ \downarrow & & \downarrow \\ \mathrm{CH}^{\mathrm{d}}(\mathrm{X}-\mathrm{Y}) & \longrightarrow & \mathbf{\Gamma}_*\mathrm{H}^{2\mathrm{d}}_*(\mathrm{X}-\mathrm{Y})(\mathrm{d}) \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

Here the vertical sequences are exact.

By duality, one has

$\mathrm{H}^{2\mathrm{d}}_{*,\mathrm{Y}}(\mathrm{X})(\mathrm{d}) \cong \mathrm{H}^{2\mathrm{d}-2}_*(\mathrm{Y})(\mathrm{d}-1)$.

Assume conjectures hold for $X - Y$. Induction hypothesis for

$\mathrm{CH}^{\mathrm{d}}(\mathrm{Y}) \longrightarrow \mathbf{\Gamma}_* \mathrm{H}_*^{2\mathrm{d}-2}(\mathrm{Y})(\mathrm{d}-1)$ implies
conjectures.

□

Take general strict normalcrossing divisors $D_1, \cdots, D_{2d}$ on X excluding Y.

Since $X-Y$ is affine, let A denote $\mathbf{\Gamma}(X-Y, \mathcal{O})$, which is geometrically regular commutative $k$algebra of finite type. Thus
$X - Y = \mathrm{Spec\ A}$. Let $I_1, \cdots, I_{2d}$ be ideals of A such that
$V(I_1) = D_1, \cdots, V(I_{2d}) = D_1 \cap \cdots \cap D_{2d}$.
By Nöther's normalization lemma,
one obtains algebraically independent elements $x_1, \cdots, x_{2d}$ of A such that

(a): A is integral over $k[x_1, \cdots, x_{2d}]$.
(b): $\mathrm{I}_i \cap\ k[x_1, \cdots, x_{2d}] = (x_1, \cdots, x_\mathrm{i})$.

Let $K = k(x_1, \cdots, x_{2d})$ and $L = Q(A)$ the total fraction field of A. Let $K \subset L$ be a finite extension and M the smallest quasi-Galois extension of L/K .

Take the integral closure B of A over $k[X]$ in M . Let $G = \mathrm{G}al(M/K)$.

Lemma 19. *The action of G on M keeps B to be invariant.*

Proof. Since A is regular; hence normal, A coincides with the integral closure of $k[x_1, \cdots, x_{2d}]$
in L .
Let $a \in A$ with the minimal polynomial $h$and let $\sigma \in G$. Then
$h^\sigma = h$.
Take any $b \in B$, which satisfies the minimal polynomial $g(b) = 0$ with
coefficients in A . Thus g^σ has its coefficients in A. Hence $g^\sigma(b^\sigma) = 0$, which implies
b is an integral element of M. Therefore $b^\sigma \in B$.

□

Let K' be the field of the invariants of M by G and C the integral closure of $k[x_1, \cdots, x_{2d}]$. . Note that K' is a radical extension of K of finite degree.

Lemma 20. : *(i)* $B^G = C$
: *(ii)* $\mathrm{SpecC} \to \mathrm{Speck}[\mathrm{x}_1, \cdots, \mathrm{x}_{2\mathrm{d}}]$ *is a finite, surjective and radical morphism, which is a universal homeomorphism.*

Proof. Since $M \supset B \supset A \supset C \supset k[x_1, \cdots, x_{2d}]$, one has $K' = M^G \supset B^G \supset A^G \supset C^G = C$.

Since B is a finite $k[x_1, \cdots, x_{2d}]$module and $k[x_1, \cdots, x_{2d}]$ is a Nötherian ring, every submodule of B is a finite $k[x_1, \cdots, x_{2d}]$ module.

Every element of B^G is integral over C. Hence $B^G = C$ since C is integrally closed.

$\phi : \mathrm{Spec\ C} \to \mathrm{Spec\ k[x_1, \cdots, x_{2d}]}$

For every point x of Spec C one has $\kappa(x)$ is a radical extension of $\kappa(\phi(x))$; thus $\phi : \mathrm{Spec\ C} \to \mathrm{Spec\ k[x_1, \cdots, x_{2d}]}$ is a radical morphism. It is clear that the mophism is finite and surjective; hence a universal homeomorphism.

□

Let $\mathcal{F}$ be a suitable smooth sheaf on X.

One has a trace map : $tr_{L/K'} : L \to K'$, which naturally extends to a map $\mathrm{tr_{A/C}} : \mathrm{A} \to \mathrm{C}$. If $f \in C$, one further has a map $tr_{A[\frac{1}{f}]/C[\frac{1}{f}]}$: $A[\frac{1}{f}] \to C[\frac{1}{f}]$ and a cohomological map

$$tr_{A[\frac{1}{f}]/C[\frac{1}{f}]} : \mathrm{H^j}\left(\mathrm{Spec\ A}[\frac{1}{\mathrm{f}}], \mathcal{F}\right) \to \mathrm{H^j}\left(\mathrm{Spec\ C}[\frac{1}{\mathrm{f}}], \mathcal{F}\right).$$

If $f \in k[x_1, \cdots, x_{2d}]$, $\phi : \mathrm{Spec\,C}[\frac{1}{\mathrm{f}}] \to \mathrm{Spec\,k}[\mathrm{x_1, \cdots, x_{2d}}, \frac{1}{\mathrm{f}}]$ is a finite, surjective and radical morphism; hence a universal homeomorphism. Thus one has an isomorphism

$$\mathrm{H^j}\left(\mathrm{Spec\ C}[\frac{1}{\mathrm{f}}], \mathcal{F}\right) \to \mathrm{H^j}\left(\mathrm{Spec\ k[x_1, \cdots, x_{2d}}, \frac{1}{\mathrm{f}}], \mathcal{F}\right).$$

Let $H = \mathrm{G}al(M/L)$.

Lemma 21. *Assume that* $\dim \mathrm{Spec\,A} = \mathrm{n} = 2\mathrm{d} \geq 4$. *Let* $f_1, \cdots, f_n$ *be* k*-linear combinations of* $x_1, \cdots, x_n$ *in* $k[x_1, \cdots, x_n]$. *Assume that the intersection locus* $V(f_1, \cdots, f_n)$ *is void in* $\mathrm{Spec\ k[x_1, \cdots, x_n]}$, *i.e., the hyperplanes intersect one point in infinity. One obtains* $\mathbf{\Gamma}_* \mathrm{H^n}\left(\mathrm{Spec\ A}[\frac{1}{\mathrm{f_1 \cdots f_n}}], \mathcal{F}\right) = 0$.

Proof. If $n > 2$, by the affine vanishing theorem, one has

$$\mathrm{H^{2n-1}}\left(\mathrm{Spec\ A}, \mathcal{F}\right) = 0, \mathrm{H^{2n-2}}\left(\mathrm{Spec\ A}, \mathcal{F}\right) = 0.$$

One has a Čech Spectral sequence:

$$E_1^{p,q} = \oplus_{|I|=p+1} \mathrm{H^q}\left(\cap_{\mathrm{i} \in \mathrm{I}} \mathrm{Spec\ A}[\frac{1}{\mathrm{f_i}}], \mathcal{F}\right) \Rightarrow \mathrm{H^{p+q}}\left(\mathrm{Spec\ A} - \mathrm{V(f_1, \cdots, f_n)}, \mathcal{F}\right).$$

Note that $E_1^{pq} = 0$ for $q > n$ by affine vanishing theorem and that $p > n - 1$ by Čech cohomology theory. One obtains

$$E_\infty^{n-1,n} = \mathrm{H^{2n-1}}\left(\mathrm{Spec\ A} - \mathrm{V(f_1, \cdots, f_n)}, \mathcal{F}\right) = 0,$$

$$E_2^{n-2,n} = E_\infty^{n-2,n} = Gr^{n-2}\mathrm{H}^{2n-2}((\mathrm{Spec\ A}, \mathcal{F}) = 0,$$

since $E_2^{n-4,n+1} = E_2^{n,n-1} = 0$. Note that there exists the following exact sequence:

$$\mathbf{\Gamma}_*\mathrm{H}_*^{2d-2}(\mathrm{V}(\mathrm{f_{i_k}}), \mathcal{F})(\mathrm{d}-1) \longrightarrow$$

$$\mathbf{\Gamma}_*\mathrm{H}_*^{2d}(\mathrm{Spec\ A}[\frac{1}{\mathrm{f_{i_1}\cdots f_{i_{k-1}}}}], \mathcal{F})(\mathrm{d}) \longrightarrow \mathbf{\Gamma}_*\mathrm{H}_*^{2d}(\mathrm{Spec\ A}[\frac{1}{\mathrm{f_{i_1}\cdots f_{i_k}}}], \mathcal{F})(\mathrm{d}) \longrightarrow 0.$$

Thus,

$$\oplus_{|I|=n-2}\mathbf{\Gamma}_*\mathrm{H}^n\left(\left(\mathrm{Spec\ A}[\frac{1}{\prod_{\mathrm{i\in I}}\mathrm{f_i}}], \mathcal{F}\right) \to \oplus_{|\mathrm{I}|=\mathrm{n}-1}\mathbf{\Gamma}_*\mathrm{H}^n\left(\left(\mathrm{Spec\ A}[\frac{1}{\prod_{\mathrm{i\in I}}\mathrm{f_i}}], \mathcal{F}\right)\right.\right.$$

and

$$\oplus_{|I|=n-1}\mathbf{\Gamma}_*\mathrm{H}^n\left(\left(\mathrm{Spec\ A}[\frac{1}{\prod_{\mathrm{i\in I}}\mathrm{f_i}}], \mathcal{F}\right) \to \oplus_{|\mathrm{I}|=\mathrm{n}}\mathbf{\Gamma}_*\mathrm{H}^n\left(\left(\mathrm{Spec\ A}[\frac{1}{\prod_{\mathrm{i\in I}}\mathrm{f_i}}], \mathcal{F}\right)\right.\right.$$

are surjections, respectively.

Using $E_2^{n-3,n+1} = E_2^{n+1,n-1} = E_1^{n,n} = 0$, one has

$$E_3^{n-1,n} = \mathrm{H}\left(\mathrm{E}_2^{n-3,n+1} \to \mathrm{E}_2^{n-1,n} \to \mathrm{E}_2^{n+1,n-1}\right) =$$

$$E_2^{n-1,n} = \mathrm{H}\left(\mathrm{E}_1^{n-2,n} \to \mathrm{E}_1^{n-1,n} \to \mathrm{E}_1^{n,n}\right) = \mathrm{E}_\infty^{n-1,n} = 0.$$

Hence, the homomorphism

$$E_1^{n-2,n} \to E_1^{n-1,n}$$

is a surjection.

Applying $E_2^{n-4,n+1} = E_2^{n,n-1} = 0$, one has

$$E_3^{n-2,n} = \mathrm{H}\left(\mathrm{E}_2^{n-4,n+1} \to \mathrm{E}_2^{n-2,n} \to \mathrm{E}_2^{n,n-1}\right) =$$

$$E_2^{n-2,n} = \mathrm{H}\left(\mathrm{E}_1^{n-3,n} \to \mathrm{E}_1^{n-2,n} \to \mathrm{E}_1^{n-1,n}\right) = \mathrm{E}_\infty^{n-2,n} = 0.$$

Moreover, since

$$E_2^{n-2,n} = \mathrm{H}\left(\oplus_{|\mathrm{I}|=\mathrm{n}-2}\mathrm{H}^n(\mathrm{Spec\ A}[\frac{1}{\prod_{\mathrm{i\in I}}\mathrm{f_i}}], \mathcal{F}) \longrightarrow\right.$$

$$\left.\oplus_{|I|=n-1}\mathrm{H}^n(\mathrm{Spec\ A}[\frac{1}{\prod_{\mathrm{i\in I}}\mathrm{f_i}}], \mathcal{F}) \to \oplus_{|\mathrm{I}|=\mathrm{n}}\mathrm{H}^n(\mathrm{Spec\ A}[\frac{1}{\prod_{\mathrm{i\in I}}\mathrm{f_i}}], \mathcal{F})\right) = 0$$

and the homomorphism

$$\oplus_{|I|=n-2}\mathbf{\Gamma}_*\mathrm{H}^n(\mathrm{Spec\ A}[\frac{1}{\prod_{\mathrm{i\in I}}\mathrm{f_i}}], \mathcal{F}) \to \oplus_{|\mathrm{I}|=\mathrm{n}-1}\mathbf{\Gamma}_*\mathrm{H}^n(\mathrm{Spec\ A}[\frac{1}{\prod_{\mathrm{i\in I}}\mathrm{f_i}}], \mathcal{F})$$

is surjective and a functor $\mathbf{\Gamma}_*$ is exact, it implies that

$$\oplus_{|I|=n-1}\mathbf{\Gamma}_*\mathrm{H}^n(\mathrm{Spec\ A}[\frac{1}{\prod_{i\in I}f_i}],\mathcal{F}) \to \oplus_{|I|=n}\mathbf{\Gamma}_*\mathrm{H}^n(\mathrm{Spec\ A}[\frac{1}{\prod_{i\in I}f_i}],\mathcal{F})$$

is a zero map.

On the other hand,

$$\oplus_{|I|=n-1}\mathbf{\Gamma}_*\mathrm{H}^n(\mathrm{Spec\ A}[\frac{1}{\prod_{i\in I}f_i}],\mathcal{F}) \to \oplus_{|I|=n}\mathbf{\Gamma}_*\mathrm{H}^n(\mathrm{Spec\ A}[\frac{1}{\prod_{i\in I}f_i}],\mathcal{F})$$

is a surjection. Thus one concludes that

$$\mathbf{\Gamma}_*\mathrm{H}^n(\mathrm{Spec\ A}[\frac{1}{f_1\cdots f_n}],\mathcal{F}) = \oplus_{|I|=n}\mathbf{\Gamma}_*\mathrm{H}^n(\mathrm{Spec\ A}[\frac{1}{\prod_{i\in I}f_i}],\mathcal{F}) = 0.$$

□

Theorem 13. *For $0 \leq k \leq n$ the images of*

$$\mathrm{CH}^d(\mathrm{Spec\ A}[\frac{1}{f_1\cdots f_k}]) \longrightarrow \mathbf{\Gamma}_*\mathrm{H}^{2d}(\mathrm{Spec\ A}[\frac{1}{f_1\cdots f_k}])(d)$$

generate the targets.

In particular, the image of

$$\mathrm{CH}^d(\mathrm{Spec\ A}) \longrightarrow \mathbf{\Gamma}_*\mathrm{H}^{2d}_*(\mathrm{Spec\ A})(d)$$

generates the target.

Proof. One continues to proceed by induction argument:

$\mathrm{H}^n_{V(f_1)}(\mathrm{Spec\ A},\mathcal{F}) \to \mathrm{H}^n(\mathrm{Spec\ A},\mathcal{F}) \to \mathrm{H}^n\left(\mathrm{Spec\ A}[\frac{1}{f_1}],\mathcal{F}\right)$

$\cdots$

$\mathrm{H}^n_{V(f_n)}\left(\mathrm{Spec\ A}[\frac{1}{f_1\cdots f_{n-1}}],\mathcal{F}\right) \to \mathrm{H}^n\left(\mathrm{Spec\ A}[\frac{1}{f_1\cdots f_{n-1}}],\mathcal{F}\right) \to \mathrm{H}^n\left(\mathrm{Spec\ A}[\frac{1}{f_1\cdots f_n}],\mathcal{F}\right)$

One has the following commutative diagram, whose vertical sequences are exact:

$$\begin{array}{ccc}
\mathrm{CH}^{d-1}(V(f_k)) & \longrightarrow & \mathbf{\Gamma}_*\mathrm{H}^{2d-2}_*(V(f_k))(d-1) \\
\downarrow & & \downarrow \\
\mathrm{CH}^d(\mathrm{Spec\ A}[\frac{1}{f_1\cdots f_{k-1}}]) & \longrightarrow & \mathbf{\Gamma}_*\mathrm{H}^{2d}_*(\mathrm{Spec\ A}[\frac{1}{f_1\cdots f_{k-1}}])(d) \\
\downarrow & & \downarrow \\
\mathrm{CH}^d(\mathrm{Spec\ A}[\frac{1}{f_1\cdots f_k}]) & \longrightarrow & \mathbf{\Gamma}_*\mathrm{H}^{2d}_*(\mathrm{Spec\ A}[\frac{1}{f_1\cdots f_k}])(d) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

□

In Case of Singular Varieties. Uwe Jannsen proved that the Hodge conjecture and the Tate conjecture for singular varieties are deduced by the original conjectures. For the readers convienience we explain it. For a smooth variety X of dimension d one has the Poincaré duality $\mathrm{H}_{2d}(X, i) \cong \mathrm{H}^{2d-2i}(X, d-i)$. There is no such duality in general for non smooth varieties. Fundamental classes induce a cycle map: $cl_i : Z_i(X) \longrightarrow \mathrm{H}_{2i}(X, i)$, which factors through the canonical cycle map that Fulton defines $\mathrm{CH}_i(X) \longrightarrow \mathrm{H}_{2i}(X, i)$.

The Hodge conjecture for singular varieties says that for all $i \geq 0$ the map

$$cl_i \otimes \mathbb{Q} : Z_i(X) \otimes \mathbb{Q} \longrightarrow \Gamma_* \mathrm{H}_{2i}(X, \mathbb{Q})(i) = (2\pi i)^{-i} W_{-2i} \mathrm{H}_{2i}(X, \mathbb{Q}) \cap F^{-i} \mathrm{H}_{2i}(X, \mathbb{C})$$

is surjective.

Theorem 14. *The Hodge conjecture is true for singular varieties.*

Proof. By Chow's lemma and Hironaka's resolution of singularities for a singular non complete variety X there exist $\pi : X' \to X$ a projective and sujective morphism with X' quasi-projective and smooth and $\alpha : X' \to X''$ an open immersion with X'' projective and smooth forming a diagram of varieties

$$\begin{array}{ccc} X' & \xrightarrow{\alpha} & X'' \\ \pi \downarrow & & \\ X & & \end{array}$$

$$\begin{array}{ccccc} Z_i(X'') \otimes F & \xrightarrow{\alpha^*} & Z_i(X') \otimes F & \xrightarrow{\pi^*} & Z_i(X) \otimes F \\ \downarrow cl_i & & \downarrow cl_i & & \downarrow cl_i \\ \Gamma \mathrm{H}_{2i}(X'', i) & \xrightarrow{\Gamma\alpha^*} & \Gamma W_0 \mathrm{H}_{2i}(X', i) & \xrightarrow{\Gamma\pi^*} & \Gamma W_0 \mathrm{H}_{2i}(X, i) \end{array}$$

□

Since $\mathrm{H}_{2i}(X'', i)$ is a semi-simple object, $W_0 \mathrm{H}_{2i}(X, i)$ is a direct factor of $\mathrm{H}_{2i}(X'', i)$ via $\pi_* \circ \alpha^*$ and so $\Gamma\pi_* \circ \Gamma\alpha^*$ is surjective. Thus $Z_i(X) \otimes F \to \Gamma W_0 \mathrm{H}_{2i}(X, i)$ is surjective.

12. Appendix1

Theorem 15. *Let $f_0 :: X_0 \to Y_0$ a projective morphism, $\ell \in \mathrm{H}^2(X_0, \mathbf{Q}_\ell(1))$ the first Chern class of an f_0-ample invertible sheaf and $\mathcal{F}_0$ a perverse sheaf over X_0 for $i \geq 0$. The following map is an isomorphism*

$$\ell^i : {}^p\mathrm{H}^{-i} f_* \mathcal{F}_0 \cong {}^p\mathrm{H}^i f_* \mathcal{F}_0(i)$$

Lemma 22. *It suffices to prove the Hodge conjecture in case of* $i = 2d = \dim X$*.*

Proof. By the strong Lefschetz theorem it reduces to the case $i = 2p > 2d$. Let Y be a general hyperplane section of X. By the weak Lefschetz one has an exact sequence $\mathrm{H^{i-2}(Y,\mathcal{F}) \to H^i(X,\mathcal{F}) \to H^i(X-Y,\mathcal{F}) = 0}$. The following commutative diagram completes the proof:

$$\begin{array}{ccc} \mathrm{CH^{p-1}(Y)} & \longrightarrow & \mathbf{\Gamma}_*\mathrm{H^{2p-2}(Y)(p-1)} \\ \downarrow & & \downarrow \\ \mathrm{CH^p(X)} & \longrightarrow & \mathbf{\Gamma}_*\mathrm{H^{2p}_*(X)(p)} \\ \downarrow & & \downarrow \\ \mathrm{CH^p(X-Y)} & \longrightarrow & \mathbf{\Gamma}_*\mathrm{H^{2p}_*(X-Y)(p)} \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

□

Theorem 16. *Let* $f : X \to Y$ *be an affine morphism. The functor*

$$Rf_* : D^b_c(X, \overline{\mathbf{Q}}_\ell) \to D^b_c(Y, \overline{\mathbf{Q}}_\ell)$$

is right t-exact. In particular, Let k *be an algebraically closed field and* $\mathcal{F}$ *an etale sheaf on* X*.* $\mathrm{H^i(X,\mathcal{F})} = 0$ *for* $i > \dim X$*.*

13. Appendix2

13.1. **Galois Theory.**

We consider commutative fields in the following subsection.

Definition 5. *An extension* L *of a field* k *is said to be primary if the largest algebraic separable extension of* k *in* L *coincides with* k*.*

Proposition 11. *Let* X *be a* k*-scheme. The following statements are equivalent.*

- : *(a) For every extension* K/k, $X \otimes_k K$ *is irreducible,i.e., geometrically irreducible.*
- : *(b) For every finite separable extension* K/k, $X \otimes_k K$ *is irreducible.*
- : *(c)* X *is irreducible and if* x *is a generic point,* $k(x)$ *is a primary extension of* k*.*

Proposition 12. *Let* Ω *be an algebraically closed field of* K *and all extensions of* K *subextensions of* ω*.* N *a Galois extension of a field* K*,* E *any extension of* K *and* $L = N \cap E$*. Then the fields* E *and* N *are linearly disjoint over* L*, i.e.,* $E(N) \cong E \otimes_L N$*.*

$$\mathrm{Gal}(E(N)/E) \cong \mathrm{Gal}(N/(E \cap N))$$

TABLE 1

$$\begin{array}{ccc} N & \to & E(N) \\ \uparrow & & \uparrow \\ L & \to & E \\ \uparrow & & \\ K & & \end{array}$$

Corollary 9. *For a field F such that $E \subset F \subset E(N)$, one obtains*

$$F = E(F \cap N)$$

Corollary 10. *Let E_1 and E_2 be two Galois extensions of K such that $E_1 \cap E_2 = K$. Then E_1 and E_2 are linearly disjoint over K and $K(E_1 \cup E_2)$ is a Galois extension of K.*

$$\text{Gal}(K(E_1 \cup E_2)/K) \cong \text{Gal}(E_1/K) \times \text{Gal}(E_2/K) \tag{1}$$

TABLE 2

$$\begin{array}{ccc} E_1 & \to & K(E_1 \cup E_2) \\ \uparrow & & \uparrow \\ E_1 \cap E_2 & \to & E_2 \end{array}$$

Proposition 13. *Let E_1 and E_2 be two extensions of K such that $E_1 \cap E_2 = K$ and linearly disjoint over K. Then one obtains*

$$\text{Gal}(\overline{K(E_1 \cup E_2)})/K(E_1 \cup E_2)) \cong \text{Gal}(\overline{E_1}/E_1) \times \text{Gal}(\overline{E_2}/E_2)$$

TABLE 3

$$\begin{array}{ccccc} \overline{E_1} & \to & \overline{E_1}(E_2) & \to & K(\overline{E_1} \cup \overline{E_2}) \\ \uparrow & & \uparrow & & \uparrow \\ E_1 & \to & K(E_1 \cup E_2) & \to & \overline{E_2}(E_1) \\ \uparrow & & \uparrow & & \uparrow \\ E_1 \cap E_2 & \to & E_2 & \to & \overline{E_2} \end{array}$$

Theorem 17 (Galois Correspondence). *Let L be an infinite Galois extension of F, $G = \text{Gal}(L/F)$. For every closed subgroup H of G, let L^H denote the fixed field of H. The correspondence*

$$K \mapsto \text{Gal}(L/K)$$

defined for all intermedediate field extensions $F \subset K \subset L$ is an inclusion reversing bijection between the set of all intermediate extensions K and the set of all closed subgroups of G. Its inverse is the correspondence

$$H \mapsto L^H$$

defined for all closed subgroups H of G. The extension K/F is normal if and only if $\mathrm{Gal}(L/K)$ is a normal subgroup of G and one obtains the exact sequence

$$1 \to \mathrm{Gal}(L/K) \to \mathrm{Gal}(L/F) \to \mathrm{Gal}(K/F) \to 1$$

Theorem 18. *Let K be a field of characteristic 0, n, m positive integers and let $x_1, \cdots, x_n$ be transcendental indeterminates over K.*

Table 4

$$\begin{array}{ccc}
\overline{K(x_1,\cdots,x_n)} & & \\
\uparrow & \nwarrow & \\
K(x_0,x_1,\cdots,x_n) & \longrightarrow & \overline{K(x_1,\cdots,x_m)}(x_{m+1},\cdots,x_n) \\
\uparrow & & \uparrow \\
K(x_0,x_1,\cdots,x_n)\cap\overline{K(x_1,\cdots,x_m)} & \longrightarrow & \overline{K(x_1,\cdots,x_m)} \\
\uparrow & \nearrow & \\
K(y_0,x_1,\cdots,x_m) & &
\end{array}$$

Then one has

$$\mathrm{Gal}\left(\overline{K(x_1,\cdots,x_m)}(x_{m+1},\cdots,x_n)/K(x_0,x_1,\cdots,x_)\right) \cong$$

$$\mathrm{Gal}\left(\overline{K(x_1,\cdots,x_m)}/K(x_0,x_1,\cdots,x_n)\cap\overline{K(x_1,\cdots,x_m)}\right)$$

Table 5

$$\begin{array}{ccc}
\mathrm{Gal}\left(\overline{K(x_1,\cdots,x_n)}/K(x_0,x_1,\cdots,x_n)\right) & \to & \mathrm{Gal}\left(\overline{K(x_1,\cdots,x_m)}/K(x_0,x_1,\cdots,x_n)\cap\overline{K(x_1,\cdots,x_m)}\right) \\
& \searrow & \downarrow \\
& & \mathrm{Gal}\left(\overline{K(x_1,\cdots,x_m)}/K(y_0,x_1,\cdots,x_m)\right)
\end{array}$$

Conversely, one can construct x_0 and y_0 in $\overline{K(x_1,\cdots,x_n)}$ and $\overline{K(x_1,\cdots,x_m)}$, respectively. Profinite groups have cohomological dimensions and let cd denote the cohomological dimension. Then cdKer $(\mathrm{Gal}(/K(x_1,\cdots,x_n)) \to \mathrm{Gal}(/K))$ *coincides with the trancendental dimension of an extension* $\overline{K}(x_1,\cdots,x_n)/\overline{K}$ *and we have*

$$\mathrm{Hom}(\mathrm{Spec}(K(x_0,x_1,\cdots,x_n)),\mathrm{Spec}(K(y_0,x_1,\cdots,x_n)))$$

$$\cong \mathrm{Hom}^{opencont}_{\mathrm{Gal}(\overline{K}/K)}(\mathrm{Gal}\left(\overline{K(x_1,\cdots,x_n)}/K(x_0,x_1,\cdots,x_n)\right),\mathrm{Gal}\left(\overline{K(x_1,\cdots,x_m)}/K(y_0,x_1,\cdots,x_m)\right)).$$

Theorem 19. *Let K be an algebraic number field and X a complete normal variety of general type over K. $X(K)$ is not dense in X.*

Proof. Assume the conclusion is not true. One will prove the theorem by absurdity. Let $U_\beta \subset U_\alpha$ be open subvarieties of X for a partial order $\alpha \geq \beta$.

TABLE 6

$$\begin{array}{ccccccccc} 1 & \to & \pi_1(\overline{U_\alpha}) & \to & \pi_1(U_\alpha) & \to & \pi_1(K,\overline{K}) & \to & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \to & \pi_1(\overline{U_\beta}) & \to & \pi_1(U_\beta) & \to & \pi_1(K,\overline{K}) & \to & 1 \end{array}$$

Here the vertical upward arrows are surjective homomorphisms. □

Let π be a profinite group. Let $C(\pi)$ be a category of the finite sets on which π acts continuously.

Proposition 14 (SGA1). *The category* Pro$-C(\pi)$ *of the pro-objects of $C(\pi)$ is canonically equivalent to the category $C'(\pi)$ of the profinite spaces on which π acts continuously.*

Let S be a locally noetherian connected scheme. Let $a : \Omega \to S$ be a geometric poit of S, where Ω is an algebraically closed field. Let C be a category of etale coverings of S. Let F be a functor from C to the category of the sets. For an etale covering X/S, $F(X)$ is the set of geometric points over a.

Proposition 15. *Let $f : X \to S$ be an etale surjection. Then one has* $\mathrm{Outmon}^{open}(\pi_1(X) \to \pi_1(S)) \cong \mathrm{Cov}(X,S)$.

Proof. □

Lemma 23. *Canonical homomorphisms* $\mathrm{Aut}(U_\alpha) \to \mathrm{Out}(\pi_1(U_\alpha))$ *are epimorphisms with splitting.*

Proof. There is an identification of universal coverings of U_α.

□

Continuity of proof. One has $\mathrm{Out}(\lim_{\leftarrow}\pi_1(\overline{U_\alpha})) = \mathrm{Out}(\mathrm{Gal}(\overline{R(\overline{X})}/R(\overline{X}))) = \mathrm{Bir}(\overline{X})$.

Since $\mathrm{Out}(\pi_1(\overline{U}_\alpha)$ is identified as a subgroup of $\mathrm{Bir}(\overline{X})$, there are only finitely many such groups.

Consider

$$H^1(\pi_1(K,\overline{K}), \mathrm{Out}(\pi_1(\overline{U}_\alpha)) \tag{2}$$

Replacing K by a finite extension F of K, one obtains

$$H^1(\pi_1(F,\overline{K}), \mathrm{Out}(\pi_1(\overline{U}_\alpha)) = 1 \tag{3}$$

One can find an extension F/K such that for all

$$H^1(\pi_1(F,\overline{K}), \mathrm{Out}(\pi_1(\overline{U}_\alpha)) = 1. \tag{4}$$

Hence $\pi_1(U_\beta) = \pi_1(\overline{U_\beta}) \times \pi_1(F,\overline{K})$ for all β. Therefore, $\mathrm{Gal}(\overline{R(X)}/R(X)) = \mathrm{Gal}(\overline{R(\overline{X})}/R(\overline{X})) \times \mathrm{Gal}(\overline{F}/F)$. It is absurd.

□

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