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Bifurcation currents for the family of symmetric products of quadratic polynomials

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In this note, we consider the dynamics of a family of regular polynomial endomorphisms of $\mathbb{C}^2$, obtained as the symmetric products of quadratic polynomials of one variable. The support of the bifurcation current of the family is investigated.

1 Introduction

We will consider the family of regular polynomial endomorphisms H_c of $\mathbb{C}^2$ of the form :

$$H_c(x,y) = (x^2 - 2y + 2c, y^2 + cx^2 - 2cy + c^2), \quad c \in \mathbb{C}.$$

A polynomial map f of $\mathbb{C}^k$ of degree d is called *regular* if its homogeneous part f_h of degree d satisfies $f_h^{-1}(0) = \{0\}$. Any regular polynomial map extends to an analytic map of $\mathbb{P}^k$. Let Π denote the hyperplane at ∞, which is isomorphic to $\mathbb{P}^{k-1}$. In case $k = 2$, Π is isomorphic to the Riemann sphere $\overline{\mathbb{C}}$. Put $f_\Pi = f|_\Pi$. As for H_c, since $H_{c,h}(x,y) = (x^2, y^2 + cx^2)$, it is regular and $H_{c,\Pi}$ is the quadratic polynomial $p_c(\zeta) = \zeta^2 + c$.

The map H_c is obtained as the *symmetric product* of the quadratic map $p_c(z) = z^2 + c$ of one variable (see Morosawa et. al. [MNTU]). That is, put $(x,y) = \psi(s,t) = (s+t, st)$ and $P_c(s,t) = (p_c(s), p_c(t))$. Then it follows

$$H_c \circ \psi(s,t) = (s^2 + c + t^2 + c, (s^2 + c)(t^2 + c)) = \psi \circ P_c(s,t).$$

The critical set $\mathcal{C} = \mathcal{C}(H_c)$ of H_c is directly calculated and is equal to $\{xy = 0\}$. For general symmetric products, we have

Lemma 1.1. $\mathcal{C}(H_c) = \psi((\mathcal{C}(p_c) \times \mathbb{C}) \cup (\mathbb{C} \times \mathcal{C}(p_c)) \cup \{\dfrac{p_c(s) - p_c(t)}{s - t} = 0\})$.

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proof. Taking the jacobians of both sides of the equality : $H_c \circ \psi(s,t) = (p_c(s) + p_c(t), p_c(s)p_c(t))$, we have

$$\begin{aligned} Jac(H_c)(\psi(s,t))Jac(\psi)(s,t) &= det \begin{pmatrix} p_c'(s) & p_c'(t) \\ p_c'(s)p_c(t) & p_c(s)p_c'(t) \end{pmatrix} \\ &= p_c'(s)p_c'(t)(p_c(s) - p_c(t)). \end{aligned}$$

Since $Jac(\psi)(s,t) = s - t$, it follows

$$Jac(H_c)(\psi(s,t)) = p_c'(s)p_c'(t)\frac{p_c(s) - p_c(t)}{s - t}.$$

This completes the proof. $\square$

Let $G_f(z) = \lim_{n\to\infty} d^{-n} \log^+ |f^n(x)|$ be the *Green function* for a regular polynomial endomorphism f of $\mathbb{C}^k$ of degree d. It is a continuous plurisubharmonic function on $\mathbb{C}^k$ and defines a positive closed $(1,1)$-current $T := dd^c G_f$. Then $\mu := T^k$ is well defined and gives a positive invariant measure. In fact, it is the ergodic measure of maximal entropy for f.

Put $G_c = G_{H_c}$. Note that $G_{P_c}(s,t) = max\,(G_{p_c}(s), G_{p_c}(t))$. Let ϕ_0 and ϕ_1 be the two branches of the inverse of the map ψ. The relation $(x,y) = \psi(s,t)$ implies s and t are the roots of the quadratic equation $z^2 - xz + y = 0$. Hence, if $\phi_0(x,y) = (s,t)$, then $\phi_1(x,y) = (t,s)$. By the semiconjugacy $H_c \circ \psi = \psi \circ P_c$, we have

$$G_c(x,y) = G_{P_c} \circ \phi_0(x,y) = G_{P_c} \circ \phi_1(x,y).$$

2 Bifurcation current

We will consider the bifurcation current of the family $\{H_c\}$. First we calculate the sum of Lyapunov exponents for H_c. The *sum of Lyapunov exponents* $\Lambda(f)$ of a regular polynomial endomorphism f of $\mathbb{C}^k$ is defined by

$$\Lambda(f) = \Lambda(f,\mu) = \lim_{n\to\infty} \frac{1}{n} \log |det\, Df^n(x)|,$$

for μ-a.e. $x \in \mathbb{C}^k$. By the ergodicity of μ, it follows

$$\Lambda(f) = \int \log |det\, Df|\, \mu.$$

For a polynomial map p on $\mathbb{C}$ of degree d, $\Lambda(p)$ is as follows :

$$\Lambda(p) = \log d + \sum G_p(c_j), \tag{1}$$

where c_j are the finite critical points of p. See Mañé [Ma], Manning [Mn] or Przytycki [P]. The *bifurcation current* T_{bif} of a holomorphic family $\{f_\lambda\}_{\lambda \in X}$ of regular polynomial endomorphisms is given by $T_{bif} := dd^c_\lambda \Lambda(f_\lambda)$.

We will use the following result due to Bedford and Jonsson [BJ]. The *critical measure* μ_{crit} for f is defined by $\mu_{crit} := [\mathcal{C}] \wedge T^{k-1}$. Here $[\mathcal{C}]$ is the current of integration along $\mathcal{C}$.

Lemma 2.1. *([BJ], Theorem 3.2) The sum of Lyapunov exponents $\Lambda(f)$ of a regular polynomial endomorphism f of $\mathbb{C}^k$ of degree d is expressed by*

$$\Lambda(f) = \log d + \Lambda(f_\Pi) + \int G_f \, \mu_{crit}.$$

Here is the main theorem in this section.

Theorem 2.1. $\Lambda(H_c) = 2\Lambda(p_c)$.

proof. If we put $\mathcal{C}_1 = \{y = 0\}$ and $\mathcal{C}_2 = \{x = 0\}$, we have $[\mathcal{C}] = [\mathcal{C}_1] + [\mathcal{C}_2]$.

First consider $[\mathcal{C}_1] \wedge T = dd^c_x G_c(x, 0)$. Since $\phi_j(x, 0)$ are either $(x, 0)$ or $(0, x)$, it follows

$$G_c(x, 0) = G_{P_c}(x, 0) = max\left(G_{p_c}(x), G_{p_c}(0)\right).$$

Thus,

$$\begin{aligned}
[\mathcal{C}_1] \wedge T &= dd^c_x G_c(x, 0) \\
&= dd^c_x \, max\left(G_{p_c}(x), G_{p_c}(0)\right) \\
&= \varphi_c^* dd^c_t \, max\left(\log^+ |t|, G_{p_c}(0)\right) \\
&:= \lim_{\epsilon \searrow G_{p_c}(0)} \varphi_c^* dd^c_t \, max\left(\log^+ |t|, \epsilon\right).
\end{aligned}$$

Here we give some remarks on the right hand side of the above equality. Since the Böttcher coordinate φ_c of p_c is invertible in the region $G_{p_c}(x) > G_{p_c}(0)$, the pull-backs $\varphi_c^* dd^c_t \, max\left(\log^+ |t|, \epsilon\right)$ are well defined for $\epsilon > G_{p_c}(0)$. Then the right hand side can be regarded as their weak limit as $\epsilon \searrow G_{p_c}(0)$ and is supported on the critical equipotential $G_{p_c}(x) = G_{p_c}(0)$. Hence $\int G_c [\mathcal{C}_1] \wedge T = G_{p_c}(0)$. See Appendix.

Next consider $[\mathcal{C}_2] \wedge T$. Since $\phi_j(0, y)$ are either $(\sqrt{-y}, -\sqrt{-y})$ or $(-\sqrt{-y}, \sqrt{-y})$, it follows

$$\begin{aligned}
G_c(0, y) &= G_{P_c}(\sqrt{-y}, -\sqrt{-y}) \\
&= max\left(G_{p_c}(\sqrt{-y}), G_{p_c}(-\sqrt{-y})\right) \\
&= G_{p_c}(\sqrt{-y}) \\
&= \frac{1}{2} G_{p_c}(-y + c).
\end{aligned}$$

Thus $[\mathcal{C}_2] \wedge T = dd^c_y G_c(0, y) = dd^c_y G_{p_c}(-y + c)$. Since G_{p_c} is harmonic in the region $G_{p_c} > 0$, the support of $dd^c G_{p_c}$ is included in the region $G_{p_c} = 0$. Thus $G_c = 0$ on the support of $[\mathcal{C}_2] \wedge T$. Hence it follows $\int G_c\, [\mathcal{C}_2] \wedge T = 0$.

Now we have $\int G_c\, \mu_{crit} = \int G_c[\mathcal{C}] \wedge T = G_{p_c}(0)$. By Lemma 2.1, we conclude that

$$\Lambda(H_c) = \log 2 + \Lambda(H_{c,\Pi}) + G_{p_c}(0) = \log 2 + \Lambda(p_c) + G_{p_c}(0).$$

From (1), we have $\Lambda(p_c) = \log 2 + G_{p_c}(0)$, hence we get the conclusion. □

As a corollary, we have a very natural conclusion.

Corollary 2.1. *The support of T_{bif} coincides with the boundary of the Mandelbrot set.*

In fact, DeMarco [De2] has shown that the support of the bifurcation current for a holomorphic family of rational maps on $\overline{\mathbb{C}}$ is equal to the bifurcation locus of the family. In particular, for the family of quadratic polynomials $\{p_c; c \in \mathbb{C}\}$, the support of the bifurcation current is equal to the boundary of the Mandelbrot set.

Bassanelli and Berteloot [BB] have investigated the bifurcation currents of a holomorphic family $\{f_\lambda\}_{\lambda \in X}$ of holomorphic maps on $\mathbb{P}^k$ and obtained the following.

Proposition 2.1. *([BB], Theorem 2.2) $\Lambda(f_\lambda)$ is pluriharmonic if the repelling cycles of f_λ move holomorphically on X.*

This implies the support of the bifurcation current is included in the complement of the subset where the repelling cycles move holomorphically. Let us check this property for our family. Suppose $(x, y) = \psi(s, t)$ is a periodic point of H_c of period n. Then

$$\psi(s, t) = (x, y) = H^n_c(x, y) = \psi \circ P^n_c(s, t) = \psi(p^n_c(s), p^n_c(t)).$$

Hence it follows $s = p^n_c(s), t = p^n_c(t)$ or $s = p^n_c(t), t = p^n_c(s)$. Taking the differentials of the semiconjugacy $H^n_c \circ \psi = \psi \circ P^n_c$, we have $DH^n_c \cdot D\psi = D\psi \cdot DP^n_c$. If $s \neq t$, then the multipliers of DH^n_c coincide with those of DP^n_c, that is, $(p^n_c)'(s)$ and $(p^n_c)'(t)$. If $s = t$, then $p^n_c(s) = s$. Although $D\psi$ is not invertible, we can also show that the multipliers of DH^n_c coincide with those of DP^n_c by approximating the point (s, s) by (s_n, t_n) with $s_n \neq t_n$. Thus we have

Lemma 2.2. *Any periodic point (x, y) of H_c is expressed as $(x, y) = \psi(s, t)$ with s and t periodic for p_c. The absolute values of the multipliers of (x, y) coincide with those of s and t. In particular, (x, y) is repelling if and only if both s and t are repelling.*

As a corollary, we get a result weaker than Corollary 2.1.

Corollary 2.2. *The support of the bifurcation current for our family is included in the boundary of the Mandelbrot set.*

proof. This follows from the fact that, in the complement of the boundary of the Mandelbrot set, all the repelling cycles move holomorphically. □

3 Uniformly laminar structure

Let $\psi_c = \varphi_c^{-1}$ be the inverse Böttcher coordinate of p_c. Then $\Psi_c(u,v) := (\psi_c(u) + \frac{v}{u}, \frac{v}{u}\psi_c(u))$ is the inverse Böttcher coordinate of H_c defined on $W^s(J_\Pi, H_{c,h}) \cap A_{0,H_c}$. If $K_c = K(p_c)$ is connected, ψ_c is a conformal isomorphism $\mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} \setminus K_c$.

If we put $\Theta : J_\Pi \times (\mathbb{C} \setminus \overline{\mathbb{D}_R}) \to W^s(J_\Pi, H_{c,h}) \cap A_{0,H_c}$ by $\Theta(\zeta, t) = (t, \zeta t)$, we have

$$\begin{aligned} H_c \circ \Psi_c \circ \Theta(\zeta, t) &= H_c \circ \Psi_c(t, \zeta t) \\ &= \Psi_c(t^2, (\zeta^2 + c)t^2) \\ &= \Psi_c \circ \Theta(H_{c,h}(\zeta), t^2). \end{aligned}$$

Thus $\Psi_c \circ \Theta(\zeta, \cdot)$ gives a parametrization of local stable disks at $\zeta \in J_\Pi$. Applying Lemma 3.4 in [Du] to the map $\gamma_\zeta = \Psi_c \circ \Theta(\zeta, \cdot)$ with $\pi = \pi_x$, this turns out to give a Riemann surface lamination on $W^s(J_\Pi) \cap A_{0,H_c}$. It also gives a uniformly laminar structure to the current $T = dd^c G_c$ restricted to $W^s(J_\Pi) \cap A_{0,H_c}$. See [BJ]. Here we do not necessarily need their assumption that H_c is uniformly expanding on J_Π, because we have a global inverse Böttcher coordinate.

4 Appendix

Let Δ be the Laplacian on $\mathbb{R}^2 = \mathbb{C}$ and $G(z) = max(\log^+|z|, \epsilon)$ for $\epsilon > 0$. We will show the following.

Proposition 4.1. $< \Delta G, \varphi >= \int_0^{2\pi} \varphi(e^{\epsilon + i\theta}) d\theta$ *for* $\varphi \in C_0^\infty(\mathbb{R}^2)$.

Corollary 4.1. *The support of the current* $dd^c G = \frac{1}{2\pi}\Delta G dx \wedge dy$ *is equal to the circle* $|z| = e^\epsilon$.

proof. By the polar coordinate, $\Delta = \dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial \theta^2}$. Thus it follows

$$\begin{aligned} < \Delta G, \varphi > &= < G, \Delta\varphi > \\ &= \int_0^{2\pi}\int_0^{e^\epsilon} G\cdot(\varphi_{rr} + r^{-1}\varphi_r + r^{-2}\varphi_{\theta\theta}) r dr d\theta \\ &\quad + \int_0^{2\pi}\int_{e^\epsilon}^{\infty} G\cdot(\varphi_{rr} + r^{-1}\varphi_r + r^{-2}\varphi_{\theta\theta}) r dr d\theta \\ &=: I_1 + I_2. \end{aligned}$$

Here, we have

$$\begin{aligned} I_1 &= \int_0^{2\pi}\int_0^{e^\epsilon} \epsilon(r\varphi_{rr} + \varphi_r) dr d\theta \\ &= \int_0^{2\pi} \epsilon \left[r\varphi_r\right]_0^{e^\epsilon} d\theta \\ &= \int_0^{2\pi} \epsilon e^\epsilon \varphi_r(e^{\epsilon + i\theta}) d\theta, \\ I_2 &= \int_0^{2\pi}\int_{e^\epsilon}^{\infty} \log r\cdot(r\varphi_{rr} + \varphi_r) dr d\theta \\ &= \int_0^{2\pi}\int_{e^\epsilon}^{\infty} \log r\cdot(r\varphi_r)_r dr d\theta \\ &= \int_0^{2\pi} \{[\log r\cdot(r\varphi_r)]_{e^\epsilon}^{\infty} - \int_{e^\epsilon}^{\infty} r^{-1} r\varphi_r dr\} d\theta \\ &= \int_0^{2\pi} \{-\epsilon e^\epsilon \varphi_r(e^{\epsilon+i\theta}) + \varphi(e^{\epsilon+i\theta})\} d\theta. \end{aligned}$$

Then it follows

$$< \Delta G, \varphi >= \int_0^{2\pi} \varphi(e^{\epsilon+i\theta}) d\theta.$$

This completes the proof of the proposition. □

Corollary 4.2. $\int G_c\,[\mathcal{C}_1] \wedge T = G_{p_c}(0)$.

proof. Since

$$\begin{aligned} G_c(x, 0) &= max(G_{p_c}(x), G_{p_c}(0)) \\ &= \varphi_c^* max(\log^+ r, G_{p_c}(0)) \\ &= \lim_{\epsilon \searrow G_{p_c}(0)} \varphi_c^* max(\log^+ r, \epsilon), \end{aligned}$$

we have $[\mathcal{C}_1] \wedge T = \lim_{\epsilon \searrow G_{p_c}(0)} \varphi_c^* dd^c G$. By Proposition 4.1, it follows

$$\begin{aligned}
\int G_c\, [\mathcal{C}_1] \wedge T &= \lim_{\epsilon \searrow G_{p_c}(0)} < G_c(x,0), \varphi_c^* dd^c G > \\
&= \lim_{\epsilon \searrow G_{p_c}(0)} < \varphi_{c,*} G_c(x,0), dd^c G > \\
&= \lim_{\epsilon \searrow G_{p_c}(0)} \frac{1}{2\pi} \int_0^{2\pi} max(G_{p_c}(x), G_{p_c}(0)) \circ \varphi_c^{-1}(e^{\epsilon + i\theta}) d\theta \\
&= \lim_{\epsilon \searrow G_{p_c}(0)} \frac{1}{2\pi} \int_0^{2\pi} \log^+ e^{\epsilon} d\theta \\
&= \lim_{\epsilon \searrow G_{p_c}(0)} \epsilon \\
&= G_{p_c}(0).
\end{aligned}$$

This completes the proof. □

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