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Non Commutative Geometry 1

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Abstract

In this article we shall introduce a non commutative algebraic geometry by Kontsevich and Rosenberg([Kon], [Mch]) and represent it by recently develoved theory of corings and comodules([Brz]). We restrict ourselves to the category of non commutative algebraic varieties and develope the birational geometry by infinite Galois theory of skew fields making use of profinite groups([AM],[BJ],[Breen1],[Breen2],[Gir], [SGA], [S1], [S2], [Shatz], [Se], [Zuo], [RBZL]). We apply it to non commutative varieties of general type defined later over the field of characteristic 0([Iita], [Fuj], [Kaw], [Mats], [MP], [Km3]). Main tools of classification of projective varieties([Iita], [Mum], [Vieh], [Ko1], [Zuo]) are so called characteristic $p > 0$ technic([MP], [Ko2]), [BBD], [Berth]) and weak positivity direct images of multi-power of dualizing sheaves for fibre spaces([Kaw], [Ws], [Vieh], [Nak], [Km1]) as well as Kawamata-Viehweg vanishing theorems([MP]). Instead of these tools, we make use of profinite groups.

1 Introduction:

In this section we consider the corings([Brz])that have a grouplike element g which are related to ring extensions $B \to A$. Throughout this section C denotes an A-coring. Galois corings are isomorphic to the Sweedler coring associated to a ring extension $B \to A$ induced by the existence of a grouplike element. The following theorem determines when the g-coinvariants functor is an equivalence.

Theorem 2. *Let g be a grouplike element of C, $B = A_g^{coC}$, and $G_g : M^C \to M_B$ $M \mapsto M_g^{coC}$ the g-coinvariants functor.*

1. *The following statements are equivalent:*

(i) *(C, g) is a Galois coring and A is a flat left B-module.*

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(ii) *${}_AC$ is flat and A_g is a generator in M^C.*

2. *The following statements are equivalent, too.*

   (i) *(C, g) is a Galois coring and ${}_BA$ is faithfully flat.*

   (ii) *${}_AC$ is flat and A_g is a projective generator in M^C.*

   (iii) *${}_AC$ is flat and* $\mathrm{Hom^C(A_g, -) : M^C \to M_B}$ *is an equivalence whose inverse is* $- \otimes_B A : M_B \to M^C$.

The theorem above is a restatement of one of the main results in non commutative descent theory([HS], [S1], [S2], [Km2]). In fact, for an algebra extension $B \to A$, there exists a comparison functor $- \otimes_B A : M_B \to \mathrm{Desc(A/B)}$ which to each right B-module M gives a descent datum $(M \otimes_B A, f)$ with $f : M \otimes_B A \to M \otimes_B A \otimes_B A, m \otimes a \mapsto m \otimes 1_A \otimes a$. If (C, g) is a Galois coring, then the category of right C-comodules is isomorphic to the category of descent data $\mathrm{Desc(A/B)}$. Thus if $B \to A$ is faithfully flat, then it is an effective descent morphism. Furthermore, Galois corings correspond to comparison functors that are equivalences. Note that if $B \to A$ is a faithful flat extension, then $(A \otimes_B A, 1_A \otimes_B 1_A)$ is a Galois coring. The objects in the category of corings are pairs $(C : A)$, where A is an R-algebra and C is an A-coring. A morphism between corings $(C : A)$ and $(D : B)$ is a pair of mappings $(\gamma : \alpha) : (C : A) \to (D : B)$ satisfying

1. $\alpha : A \to B$ is an algebra map. Hence D is considered to be an (A, A)-bimodule.

2. $\gamma : C \to D$ is a map of (A, A)-bimodules such that

$$\xi \circ (\gamma \otimes_A \gamma) \circ \underline{\Delta}_C = \underline{\Delta}_C \circ \gamma, \underline{\varepsilon}_D \circ \gamma = \alpha \circ \underline{\varepsilon}_C,$$

   where $\xi : D \otimes_A D \to D \otimes_B D$ is the canonical map of (A, A)-bimodules.

Since an algebra A can be considered as a trivial A-coring $(A : A)$, this category of corings contains the category of R-algebras.

Left C-comodule is defined as a left A-module M, with a coassociative and counital left C-coaction.C-morphisms between left C-comodules M, N are defined in an obvious way. Left C-comodules and their morphisms form a pre-additive category ${}^CM$.

3 Geometric View

Let k be a commutative field and A, B k-algebras. The objects of the opposite category of corings denote $\mathrm{Spec}\,(C : A)$ and a morphism between $\mathrm{Spec}\,(D : B) \to \mathrm{Spec}\,(C : A)$ denotes $\mathrm{Spec}\,(\gamma : \alpha)$. This category is said to be that of covers. Furthermore, the category ${}^{C}M$ is abelian and it is denoted $QCoh(\mathrm{Spec}\,(C:A))$. The canonical morphism $f:\mathrm{Spec}\,(B\otimes_A B:B)\to\mathrm{Spec}\,(A:A)$ defines an equivalence between abelian categories $f^*:QCoh(\mathrm{Spec}\,(A:A))\cong QCoh(\mathrm{Spec}\,(B\otimes_A B:B))$. Owing to Morita-Takeuchi theorems or Grothendieck ideas, the geometry of covers consist in $QCoh(\mathrm{Spec}\,(C:A))$.

The cover $\mathrm{Spec}\,(C:A)$ equipped with an epimorphism $A\otimes A\to C$ which is a morphism of coalgebras is said to be a space cover. A morphism in the category of space covers is defined to be a morphism as covers compatible with additional structure as space covers. Let $f=(\gamma,\alpha)$, $g=(\delta,\beta)$ be two morphisms between space covers $\mathrm{Spec}\,(C:A)\to\mathrm{Spec}\,(D:B)$. When for $x_i\otimes y_i\in\ker(A\otimes A\to C$, the following equation holds $\sum_i\alpha(x_i)\cdot\beta(y_i)-\beta(x_i)\cdot\alpha(y_i)=0$ in B, two morphisms f and g are defined to be equivalent.

Definition 4. *The category of non commutative algebraic spaces over k is the localization category with the canonical morphisms invertible of the quotient of the category of space covers by equivalence of equivalent morphisms.*

The category of separated quasi-compact schemes over k([Gir], [SGA], [GG], [HS], [Kato], [KKMS]) and the opposite category of that of k-algebras are equivalent to a full subcategory of the category of non commutative algebraic spaces over k([Kon], [Cohn]), respectively. The category of non commutative algebraic spaces over k admits finite limits. A non commutative algebraic space of the type $\mathrm{Spec}\,(A:A)$, where A is a k-algebra, is said to be an affine space. Let $N\mathbf{P}_k^{d-1}$ be the non commutative projective space over k and A a k-algebra. The set $\mathrm{Hom}(\mathrm{Spec}\,(A:A),N\mathbf{P}_k^{d-1})$ is the set of quotient modules of A^d which are locally free A-modules of dimension 1 in flat topology([SGA]). In the same way, we have the non commutative Grassmannian $NGr_k(r,d)$([Kon], [Laum]).

5 Extension of skew fields and Galois theory

Let A be an integral domain such that $xA\cap yA\neq 0$ for $x,y\in A$, which is called a right Ore domain. Let $S=R^\times$. Then the localization of A at S is a skew field $K=A_S$ and the natural homomorphism $\lambda:A\to K$ is a monomorphism. Racall that every ring with a homomorphism to a field has invariant basis number. From now on, we treat a non commutative algebraic space of the type $\mathrm{Spec}\,(C:A)$ where A is a Ore domain. Any equation of degree $n>0$, $x^n+a_1x^{n-1}+\cdots+a_n=0$ $(a_i\in K)$, has a right root in some extension of K. There exists the right algebraic closure $\overline{K}$ over K such that any equation of the type above has a right root in $\overline{K}$. A Galois extension L/K is outer if and only if the centralizer of K in L is just the centralizer of L. Let k be a commutative field of characteristic 0 and K a k-algebra of finite type, skew field. Let $\overline{K}$ be the right algebraic closure of K such that the centralizer of K in $\overline{K}$ is just the centralizer of $\overline{K}$([Cohn]). Let $(K_i)_{i\in I}$ be a family of skew fields such that

1. K_i are subfields of K,
2. K_i are k-algebras of finite type,
3. the centralizers of K_i in $\overline{K}$ are the center of $\overline{K}$.

Then the $\overline{K}/K_i$ are all outer Galois extensions, whose Galois groups are profinite groups. We need Jacobson-Bourbaki correspondence([Cohn], [BJ]): Let K be a field and $End(K)$ the endomorphism ring of the additive group K^+ with the finite topology. We have an order-reversing bijection between the subfields D of K and the closed K-subrings of the type $End_{D-}(K)$ of $End(K)$. From this, we have the following Galois connection: Let L/K be an algebraic Galois extension with Galois group G outer. Then we have a bijection between intermediate fields D, i.e., $K \subset D \subset L$ and the closed subgroups H.

6 Non commutative algebraic birational geometry

We investigate the non commutative algebraic birational geometry from the point of view of the profinite Galois groups([Gir], [Breen1], [Breen2]). Let $X \to S$ be a non commutative fibre space of algebraic spaces over $\mathrm{Spec}\,(k)$, with the generic point of the generic general fibre one of skew fields K_i which are defined in the preceding section([Mch], [RBZL]). Let $1 \to G \to E \to P \to 1$ be an extension of a profinite group P by a profinite group G associated to the non commutative fibre space $X \to S$. Hence G is a profinite group, that is one of the Galois group $Gal(\overline{K}/K_i)$. To an exact sequence $1 \to \mathrm{InnG} \to \mathrm{Aut}G \to \mathrm{Out}G \to 1$, we have an exact sequence

$$\mathrm{H}^1(P, \mathrm{Inn}G) \to \mathrm{H}^1(P, \mathrm{Aut}G) \to \mathrm{H}^1(P, \mathrm{Out}G),$$

i.e.,

$$\mathrm{Hom}(P, \mathrm{Inn}G) \to \mathrm{Hom}(P, \mathrm{Aut}G) \to \mathrm{Hom}(P, \mathrm{Out}G).$$

Here $\mathrm{Out}G$ denotes the outer automorphism group of G. A group extension is an element of $\mathrm{H}^1(P, G \to \mathrm{Aut}G)$, where $G \to \mathrm{Aut}G$ is a crossed module. We have

$$1 \to \mathrm{H}^2(P, \mathrm{Z}(G)) \to \mathrm{H}^1(P, G \to \mathrm{Aut}G) \to \mathrm{H}^1(P, \mathrm{Out}G).$$

Here $\mathrm{Z}(G)$ denotes the center of G. Assume that Out(G) is an algebraic group of countable connected components. Then the canonical representation $\rho : P \to \mathrm{Out}G$ turns out to be trivial after replacing a profinite group associated to a finite morphism $S' \to S$ in the following lemma. Furthermore assume that the extension is neutral. This assumption is satisfied since there exists a homomorphism from $1 \to G' \to G' \times P \to P \to 1$ to $1 \to G \to E \to P \to 1$, where $P \to P$ is an identity, $G' = Gal(\overline{K}/K)$.

Since we have $\mathrm{H}^2(P, \mathrm{Z}(G)) \to \mathrm{H}^1(P, G \to \mathrm{Aut}(G))$, the extension $1 \to G \to E \to P \to 1$ is given by pushing out an extension $1 \to \mathrm{Z}(G) \to E' \to P \to 1$. Hence E' is a semi-direct product $\mathrm{Z}(G) \rtimes P$, which is contained in a semi-direct product $G \rtimes P$. Thus this central extension is trivial. Therefore by pushing out this central extension, the extension $1 \to G \to E \to P \to 1$ is trivial.

Lemma 7. *There exists a homomorphism $P' \to P$ with $(P' : P) < \infty$ such that the representation $\rho : P' \to \mathrm{Out}(G)$ is trivial. Here P' denotes the absolute Galois group $\mathrm{Gal}(\overline{R(S')}/R(S'))$.*

Proof. Let A denote $\mathrm{Out}(G)$. This group A is locally algebraic([SGA]). The natural representation $\rho : P \to A$ induces $\overline{\rho} : P \to A/A^0$, where A^0 denotes the neutral component of A. There is no countable profinite group. Since A/A^0 is a countable set, $\overline{\rho}(P)$ is a finite group. Replace by P the kernel of $\overline{\rho}$. We have $\rho : P \to A^0$. Hence we have an isomorphism

$$H^1(\overline{R(S)}/R(S), A^0(\overline{R(S)}) \cong H^1(BP, A^0).$$

Let P be an A^0-torsor associated to $\rho : P \to A^0$. A^0 is algebraic (quasi-compact, faithfully flat and of finite type) over $\mathrm{Spec}\,(\mathrm{R}(S))$. Thus there exists a generically finite $S' \to S$ such that an A^0-torsor P is trivial over $\mathrm{Spec}\,(\mathrm{R}(S'))$. Hence the representation $\rho : P' \to \mathrm{Out}(G)$ is trivial. □

Thus we obtain the following result in our proof.

Theorem 8. *Let $1 \to G \to E \to P \to 1$ be an extension of a profinite group P by a profinite group G. Assume*

(a) $\mathrm{Out}(G)$, is an algebraic group with countable connected components.

(b) $E \to P$ has a section which is a group homomorphism, i.e., a neutral extension.

Then there exists a profinite group P' such that the pull-back of the extension $1 \to G \times_P P' \to E \times_P P' \to P' \to 1$ is a direct product.

Let X be a non commutative fibre space of smooth varieties over $\mathrm{Spec}\,k$. We have the canonical homomorphism $\Gamma(X, \Omega_X^{\otimes m}) \otimes \mathcal{O}_X \to \Omega_X^{\otimes m}$. Assume this homomorphism is generically epimorphism. Then it determines a map from an open of X to non commutative Grassmannian[Kon]). When this map is birational, i.e., the field defined by the generic point of X and that of the image are isomorphic, the assumption (a) above is satisfied.

Remark 9. *Let $\phi : G_1 \to G_2$ be an open continuous homomorphism of profinite groups. $\phi(G_1) \subset G_2$. Let $Z(G_2)C_{\phi(G_1)}(\phi(G_1))$ denote C. Then for a homomorphism between extensions of P by G_1 and G_2 respectively, one has homomorphisms $H^2(P, Z(G_1)) \to H^2(P, \phi(Z(G_1))) \to H^2(P, C)$. There exists an open subgroup P' of finite index of P such that $H^2(P', Z(G_2)) \to H^2(P', C)$ is injective.*

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