

# A Proof of the Hodge Conjecture

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## 1 Introduction

In this paper we show that the Hodge conjecture and a part of the Tate conjecture hold. Since it is difficult to find algebraic cycles in general, the strategy is to proceed by induction argument to vanish a certain subspace of a cohomology of an open affine subvariety of an affine variety which is obtained excluding a general hyperplane from a given variety.

## 2 In Case of Non Singular Varieties

Let  $k$  be a field with an algebraic closure  $\bar{k}$  and  $X$  a smooth geometrically irreducible variety over  $k$ . There exists the canonical cycle map for  $\ell \neq \text{char } k$

$$cl_\ell^r : CH^r(X) \longrightarrow H_{\text{et}}^{2r}(X_{\bar{k}}, \mathbf{Q}_\ell(r))$$

This image is included in the fixed part

$$H_{\text{et}}^{2r}(X_{\bar{k}}, \mathbf{Q}_\ell(r))^{G_k}$$

where  $G_k = \text{Gal}(\bar{k}/k)$ . Tate's conjecture says that if  $k$  is finitely generated as a field, the image of  $cl_\ell^r$  generates  $H_{\text{et}}^{2r}(X_{\bar{k}}, \mathbf{Q}_\ell(r))^{G_k}$ . Fix an isomorphism  $\iota : \bar{\mathbf{Q}}_\ell \rightarrow \mathbf{C}$ .

Let  $k = \mathbf{C}$ . One has the canonical cycle map

$$cl^r : CH^r(X) \longrightarrow H^{2r}(X(\mathbf{C}), \mathbf{Q}(2\pi i)^r)$$

This image is included into

$$H^{2r}(X(\mathbf{C}), \mathbf{Q}(2\pi i)^r) \cap H^{r,r}(X(\mathbf{C}))$$

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Hodge conjecture says that the image of  $cl^r$  generates  $H^{2r}(X(\mathbf{C}), \mathbf{Q}(2\pi i)^r) \cap H^{r,r}(X(\mathbf{C}))$ . Let  $U$  be a smooth quasiprojective variety over  $k$ . The images of the canonical cycle maps are

$$\begin{cases} H_{\text{et}}^{2r}(U_{\bar{k}}, \mathbf{Q}_{\ell}(r))^{G_k} & \text{for finitely generated } k \\ F^r H^{2r}(U, \mathbf{C}) \cap W_{2r} H^{2r}(U, \mathbf{Q}(r)) & \text{for } k = \mathbf{C} \end{cases}$$

Let  $U$  be a smooth quasi-projective variety over  $k$  and  $X$  a smooth projective compactification of  $U$ . One denotes by  $cl_*$  the following cycle maps  $cl_{DR}$ ,  $cl_{\ell}$ ,  $cl_H$ ;

- (a)  $\Gamma_{DR}(H_{DR}^{2r}(U)(r)) = W_0(H_{DR}^{2r}(U)(r)) \cap F^0(H_{DR}^{2r}(U)(r))$
- (b)  $\Gamma_{\ell}(H_{\ell}^{2r}(U)(r)) = H_{\ell}^{2r}(U)(r)^{G_k} \cap W_0(H_{\ell}^{2r}(U)(r))$
- (c)  $\Gamma_H(H_{\sigma}^{2r}(U)(r)) = W_0(H_{\sigma}^{2r}(U)(r) \cap F^0(H_{\sigma}^{2r}(U)(r)) \otimes \mathbf{C})$

**Lemma 1** *It suffices to prove it for a smooth affine variety over  $k$ .*

**Proof.** It is well known that it is enough to treat it in the case of  $\dim X = 2d$ . Choose a smooth irreducible hyperplane  $Y$  on  $X$ . Thus  $X - Y$  is a smooth affine variety. One obtains the following commutative diagram:

$$\begin{array}{ccc} CH^{d-1}(Y) & \longrightarrow & \Gamma_* H_{*,Y}^{2d}(X)(d) \\ \downarrow & & \downarrow \\ CH^d(X) & \longrightarrow & \Gamma_* H_*^{2d}(X)(d) \\ \downarrow & & \downarrow \\ CH^d(X - Y) & \longrightarrow & \Gamma_* H_*^{2d}(X - Y)(d) \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

Here the vertical sequences are exact. By duality, one has  $H_{*,Y}^{2d}(X)(d) \cong H_*^{2d-2}(Y)(d-1)$ . Assume conjectures hold for  $X - Y$ . Induction hypothesis for  $CH^d(Y) \longrightarrow \Gamma_* H_*^{2d-2}(Y)(d-1)$  implies conjectures. ■

Take general strict normalcrossing divisors  $D_1, \dots, D_{2d}$  on  $X$  excluding  $Y$ . Since  $X - Y$  is affine, let  $A$  denote  $\Gamma(X - Y, \mathcal{O})$ , which is geometrically regular commutative  $k$ -algebra of finite type. Thus  $X - Y = \text{Spec } A$ . Let  $I_1, \dots, I_{2d}$  be ideals of  $A$  such that  $V(I_1) = D_1, \dots, V(I_{2d}) = D_1 \cap \dots \cap D_{2d}$ . By Nöther's normalization lemma, one obtains algebraically independent elements  $x_1, \dots, x_{2d}$  of  $A$  such that

- (a)  $A$  is integral over  $k[x_1, \dots, x_{2d}]$ .
- (b)  $I_i \cap k[x_1, \dots, x_{2d}] = (x_1, \dots, x_i)$ .

Let  $K = k(x_1, \dots, x_{2d})$  and  $L = Q(A)$  the total fraction field of  $A$ . Let  $K \subset L$  be a finite extension and  $M$  the smallest quasi-Galois extension of  $L/K$ . Take the integral closure  $B$  of  $A$  over  $k[X]$  in  $M$ . Let  $G = \text{Gal}(M/K)$ .

**Lemma 2** *The action of  $G$  on  $M$  keeps  $B$  to be invariant.*

**Proof.** Since  $A$  is regular; hence normal,  $A$  coincides with the integral closure of  $k[x_1, \dots, x_{2d}]$  in  $L$ . Let  $a \in A$  with the minimal polynomial  $h$  and let  $\sigma \in G$ . Then  $h^\sigma = h$ . Take any  $b \in B$ , which satisfies the minimal polynomial  $g(b) = 0$  with coefficients in  $A$ . Thus  $g^\sigma$  has its coefficients in  $A$ . Hence  $g^\sigma(b^\sigma) = 0$ , which implies  $b$  is an integral element of  $M$ . Therefore  $b^\sigma \in B$ . ■

Let  $K'$  be the field of the invariants of  $M$  by  $G$  and  $C$  the integral closure of  $k[x_1, \dots, x_{2d}]$ . Note that  $K'$  is a radical extension of  $K$  of finite degree.

**Lemma 3** (i)  $B^G = C$

(ii)  $\text{Spec } C \rightarrow \text{Spec } k[x_1, \dots, x_{2d}]$  is a finite, surjective and radical morphism, which is a universal homeomorphism.

**Proof.** Since  $M \supset B \supset A \supset C \supset k[x_1, \dots, x_{2d}]$ , one has  $K' = M^G \supset B^G \supset A^G \supset C^G = C$ . Since  $B$  is a finite  $k[x_1, \dots, x_{2d}]$  module and  $k[x_1, \dots, x_{2d}]$  is a Nötherian ring, every submodule of  $B$  is a finite  $k[x_1, \dots, x_{2d}]$  module. Every element of  $B^G$  is integral over  $C$ . Hence  $B^G = C$  since  $C$  is integrally closed.  $\phi : \text{Spec } C \rightarrow \text{Spec } k[x_1, \dots, x_{2d}]$  For every point  $x$  of  $\text{Spec } C$  one has  $\kappa(x)$  is a radical extension of  $\kappa(\phi(x))$ ; thus  $\phi : \text{Spec } C \rightarrow \text{Spec } k[x_1, \dots, x_{2d}]$  is a radical morphism. It is clear that the morphism is finite and surjective; hence a universal homeomorphism. ■

Let  $\mathcal{F}$  be a suitable smooth sheaf on  $X$ . One has a trace map :  $\text{tr}_{L/K'} : L \rightarrow K'$ , which naturally extends to a map  $\text{tr}_{A/C} : A \rightarrow C$ . If  $f \in C$ , one further has a map  $\text{tr}_{A[\frac{1}{f}]/C[\frac{1}{f}]} : A[\frac{1}{f}] \rightarrow C[\frac{1}{f}]$  and a cohomological map

$$\text{tr}_{A[\frac{1}{f}]/C[\frac{1}{f}]} : H^j \left( \text{Spec } A[\frac{1}{f}], \mathcal{F} \right) \rightarrow H^j \left( \text{Spec } C[\frac{1}{f}], \mathcal{F} \right).$$

If  $f \in k[x_1, \dots, x_{2d}]$ ,  $\phi : \text{Spec } C[\frac{1}{f}] \rightarrow \text{Spec } k[x_1, \dots, x_{2d}, \frac{1}{f}]$  is a finite, surjective and radical morphism; hence a universal homeomorphism. Thus one has an isomorphism

$$H^j \left( \text{Spec } C[\frac{1}{f}], \mathcal{F} \right) \rightarrow H^j \left( \text{Spec } k[x_1, \dots, x_{2d}, \frac{1}{f}], \mathcal{F} \right).$$

Let  $H = \text{Gal}(M/L)$ .

**Lemma 4** Assume that  $\dim \text{Spec } A = n = 2d \geq 4$ . Let  $f_1, \dots, f_n$  be  $k$ -linear combinations of  $x_1, \dots, x_n$  in  $k[x_1, \dots, x_n]$ . Assume that the intersection locus  $V(f_1, \dots, f_n)$  is void in  $\text{Spec } k[x_1, \dots, x_n]$ , i.e., the hyperplanes intersect one point in infinity. One obtains  $\Gamma_* H^n \left( \text{Spec } A[\frac{1}{f_1 \dots f_n}], \mathcal{F} \right) = 0$ .

**Proof.** If  $n > 2$ , by the affine vanishing theorem, one has

$$H^{2n-1}(\text{Spec } A, \mathcal{F}) = 0, H^{2n-2}(\text{Spec } A, \mathcal{F}) = 0.$$

One has a Čech Spectral sequence:

$$E_1^{p,q} = \oplus_{|I|=p+1} H^q \left( \bigcap_{i \in I} \text{Spec } A \left[ \frac{1}{f_i} \right], \mathcal{F} \right) \Rightarrow H^{p+q} (\text{Spec } A - V(f_1, \dots, f_n), \mathcal{F}).$$

Note that  $E_1^{pq} = 0$  for  $q > n$  by affine vanishing theorem and that  $p > n - 1$  by Čech cohomology theory. One obtains

$$E_{\infty}^{n-1,n} = H^{2n-1} (\text{Spec } A - V(f_1, \dots, f_n), \mathcal{F}) = 0,$$

$$E_2^{n-2,n} = E_{\infty}^{n-2,n} = Gr^{n-2} H^{2n-2} ((\text{Spec } A, \mathcal{F})) = 0,$$

since  $E_2^{n-4,n+1} = E_2^{n,n-1} = 0$ . Note that there exists the following exact sequence:

$$\Gamma_* H_*^{2d-2}(V(f_{i_k}), \mathcal{F})(d-1) \longrightarrow$$

$$\Gamma_* H_*^{2d}(\text{Spec } A[\frac{1}{f_{i_1} \dots f_{i_{k-1}}}], \mathcal{F})(d) \longrightarrow \Gamma_* H_*^{2d}(\text{Spec } A[\frac{1}{f_{i_1} \dots f_{i_k}}], \mathcal{F})(d) \longrightarrow 0.$$

Thus,

$$\oplus_{|I|=n-2} \Gamma_* H^n \left( \left( \text{Spec } A \left[ \frac{1}{\prod_{i \in I} f_i} \right], \mathcal{F} \right) \rightarrow \oplus_{|I|=n-1} \Gamma_* H^n \left( \left( \text{Spec } A \left[ \frac{1}{\prod_{i \in I} f_i} \right], \mathcal{F} \right) \right)$$

and

$$\oplus_{|I|=n-1} \Gamma_* H^n \left( \left( \text{Spec } A \left[ \frac{1}{\prod_{i \in I} f_i} \right], \mathcal{F} \right) \rightarrow \oplus_{|I|=n} \Gamma_* H^n \left( \left( \text{Spec } A \left[ \frac{1}{\prod_{i \in I} f_i} \right], \mathcal{F} \right) \right)$$

are surjections, respectively. Using  $E_2^{n-3,n+1} = E_2^{n+1,n-1} = E_1^{n,n} = 0$ , one has

$$E_3^{n-1,n} = H \left( E_2^{n-3,n+1} \rightarrow E_2^{n-1,n} \rightarrow E_2^{n+1,n-1} \right) =$$

$$E_2^{n-1,n} = H \left( E_1^{n-2,n} \rightarrow E_1^{n-1,n} \rightarrow E_1^{n,n} \right) = E_{\infty}^{n-1,n} = 0.$$

Hence, the homomorphism

$$E_1^{n-2,n} \rightarrow E_1^{n-1,n}$$

is a surjection.

Applying  $E_2^{n-4,n+1} = E_2^{n,n-1} = 0$ , one has

$$E_3^{n-2,n} = H \left( E_2^{n-4,n+1} \rightarrow E_2^{n-2,n} \rightarrow E_2^{n,n-1} \right) =$$

$$E_2^{n-2,n} = H \left( E_1^{n-3,n} \rightarrow E_1^{n-2,n} \rightarrow E_1^{n-1,n} \right) = E_\infty^{n-2,n} = 0.$$

Moreover, since

$$E_2^{n-2,n} = H \left( \oplus_{|I|=n-2} H^n(\text{Spec } A[\frac{1}{\prod_{i \in I} f_i}], \mathcal{F}) \longrightarrow$$

$$\oplus_{|I|=n-1} H^n(\text{Spec } A[\frac{1}{\prod_{i \in I} f_i}], \mathcal{F}) \rightarrow \oplus_{|I|=n} H^n(\text{Spec } A[\frac{1}{\prod_{i \in I} f_i}], \mathcal{F}) \right) = 0$$

and the homomorphism

$$\oplus_{|I|=n-2} \Gamma_* H^n(\text{Spec } A[\frac{1}{\prod_{i \in I} f_i}], \mathcal{F}) \rightarrow \oplus_{|I|=n-1} \Gamma_* H^n(\text{Spec } A[\frac{1}{\prod_{i \in I} f_i}], \mathcal{F})$$

is surjective and a functor  $\Gamma_*$  is exact, it implies that

$$\oplus_{|I|=n-1} \Gamma_* H^n(\text{Spec } A[\frac{1}{\prod_{i \in I} f_i}], \mathcal{F}) \rightarrow \oplus_{|I|=n} \Gamma_* H^n(\text{Spec } A[\frac{1}{\prod_{i \in I} f_i}], \mathcal{F})$$

is a zero map.

On the other hand,

$$\oplus_{|I|=n-1} \Gamma_* H^n(\text{Spec } A[\frac{1}{\prod_{i \in I} f_i}], \mathcal{F}) \rightarrow \oplus_{|I|=n} \Gamma_* H^n(\text{Spec } A[\frac{1}{\prod_{i \in I} f_i}], \mathcal{F})$$

is a surjection. Thus one concludes that

$$\Gamma_* H^n(\text{Spec } A[\frac{1}{f_1 \cdots f_n}], \mathcal{F}) = \oplus_{|I|=n} \Gamma_* H^n(\text{Spec } A[\frac{1}{\prod_{i \in I} f_i}], \mathcal{F}) = 0.$$

■

**Theorem 5** For  $0 \leq k \leq n$  the images of

$$\text{CH}^d(\text{Spec } A[\frac{1}{f_1 \cdots f_k}]) \longrightarrow \Gamma_* H^{2d}(\text{Spec } A[\frac{1}{f_1 \cdots f_k}])^{(d)}$$

generate the targets. In particular, the image of

$$\text{CH}^d(\text{Spec } A) \longrightarrow \Gamma_* H_*^{2d}(\text{Spec } A)^{(d)}$$

generates the target.

**Proof.** One continues to proceed by induction argument:  $H_{V(f_1)}^n(\text{Spec } A, \mathcal{F}) \rightarrow H^n(\text{Spec } A, \mathcal{F}) \rightarrow H^n(\text{Spec } A[\frac{1}{f_1}], \mathcal{F})$

...

$H_{V(f_n)}^n \left( \text{Spec } A[\frac{1}{f_1 \cdots f_{n-1}}], \mathcal{F} \right) \rightarrow H^n \left( \text{Spec } A[\frac{1}{f_1 \cdots f_{n-1}}], \mathcal{F} \right) \rightarrow H^n \left( \text{Spec } A[\frac{1}{f_1 \cdots f_n}], \mathcal{F} \right)$   
 One has the following commutative diagram, whose vertical sequences are exact:

$$\begin{array}{ccc}
 CH^{d-1}(V(f_k)) & \longrightarrow & \Gamma_* H_*^{2d-2}(V(f_k))(d-1) \\
 \downarrow & & \downarrow \\
 CH^d(\text{Spec } A[\frac{1}{f_1 \cdots f_{k-1}}]) & \longrightarrow & \Gamma_* H_*^{2d}(\text{Spec } A[\frac{1}{f_1 \cdots f_{k-1}}])(d) \\
 \downarrow & & \downarrow \\
 CH^d(\text{Spec } A[\frac{1}{f_1 \cdots f_k}]) & \longrightarrow & \Gamma_* H_*^{2d}(\text{Spec } A[\frac{1}{f_1 \cdots f_k}])(d) \\
 \downarrow & & \downarrow \\
 0 & & 0
 \end{array}$$

■

## In Case of Singular Varieties

Uwe Jannsen proved that the Hodge conjecture and the Tate conjecture for singular varieties are deduced by the original conjectures. For the readers convenience we explain it. For a smooth variety  $X$  of dimension  $d$  one has the Poincaré duality  $H_{2d}(X, i) \cong H^{2d-2i}(X, d-i)$ . There is no such duality in general for non smooth varieties. Fundamental classes induce a cycle map:  $cl_i : Z_i(X) \longrightarrow H_{2i}(X, i)$ , which factors through the canonical cycle map that Fulton defines  $CH_i(X) \longrightarrow H_{2i}(X, i)$ . The Hodge conjecture for singular varieties says that for all  $i \geq 0$  the map

$$cl_i \otimes \mathbb{Q} : Z_i(X) \otimes \mathbb{Q} \longrightarrow \Gamma_* H_{2i}(X, \mathbb{Q})(i) = (2\pi i)^{-i} W_{-2i} H_{2i}(X, \mathbb{Q}) \cap F^{-i} H_{2i}(X, \mathbb{C})$$

is surjective.

**Theorem 6** *The Hodge conjecture is true for singular varieties.*

**Proof.** By Chow's lemma and Hironaka's resolution of singularities for a singular non complete variety  $X$  there exist  $\pi : X' \rightarrow X$  a projective and surjective morphism with  $X'$  quasi-projective and smooth and  $\alpha : X' \rightarrow X''$  an open immersion with  $X''$  projective and smooth forming a diagram of varieties

$$\begin{array}{ccc}
 X' & \xrightarrow{\alpha} & X'' \\
 \pi \downarrow & & \\
 X & & \\
 \\
 Z_i(X'') \otimes F & \xrightarrow{\alpha^*} & Z_i(X') \otimes F \xrightarrow{\pi^*} Z_i(X) \otimes F \\
 \downarrow cl_i & & \downarrow cl_i \quad \quad \downarrow cl_i \\
 \Gamma H_{2i}(X'', i) & \xrightarrow{\Gamma \alpha^*} & \Gamma W_0 H_{2i}(X', i) \xrightarrow{\Gamma \pi^*} \Gamma W_0 H_{2i}(X, i)
 \end{array}$$

■

Since  $H_{2i}(X'', i)$  is a semi-simple object,  $W_0 H_{2i}(X, i)$  is a direct factor of  $H_{2i}(X'', i)$  via  $\pi_* \circ \alpha^*$  and so  $\Gamma\pi_* \circ \Gamma\alpha^*$  is surjective. Thus  $Z_i(X) \otimes F \rightarrow \Gamma W_0 H_{2i}(X, i)$  is surjective.

## Appendix

**Theorem 7** *Let  $f_0 :: X_0 \rightarrow Y_0$  a projective morphism,  $\ell \in H^2(X_0, \mathbf{Q}_\ell(1))$  the first Chern class of an  $f_0$ -ample invertible sheaf and  $\mathcal{F}_0$  a perverse sheaf over  $X_0$  for  $i \geq 0$ . The following map is an isomorphism*

$$\ell^i : {}^p H^{-i} f_{*} \mathcal{F}_0 \cong {}^p H^i f_{*} \mathcal{F}_0(i)$$

**Lemma 8** *It suffices to prove the Hodge conjecture in case of  $i = 2d = \dim X$ .*

**Proof.** By the strong Lefschetz theorem it reduces to the case  $i = 2p > 2d$ . Let  $Y$  be a general hyperplane section of  $X$ . By the weak Lefschetz one has an exact sequence  $H^{i-2}(Y, \mathcal{F}) \rightarrow H^i(X, \mathcal{F}) \rightarrow H^i(X - Y, \mathcal{F}) = 0$ . The following commutative diagram completes the proof:

$$\begin{array}{ccc} CH^{p-1}(Y) & \longrightarrow & \Gamma_* H^{2p-2}(Y)(p-1) \\ \downarrow & & \downarrow \\ CH^p(X) & \longrightarrow & \Gamma_* H^{2p}(X)(p) \\ \downarrow & & \downarrow \\ CH^p(X - Y) & \longrightarrow & \Gamma_* H^{2p}(X - Y)(p) \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

■

**Theorem 9** *Let  $f : X \rightarrow Y$  be an affine morphism. The functor*

$$Rf_* : D_c^b(X, \overline{\mathbf{Q}}_\ell) \rightarrow D_c^b(Y, \overline{\mathbf{Q}}_\ell)$$

*is right  $t$ -exact. In particular, Let  $k$  be an algebraically closed field and  $\mathcal{F}$  an étale sheaf on  $X$ .  $H^i(X, \mathcal{F}) = 0$  for  $i > \dim X$ .*

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