

Galois Theory

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0 Introduction

In this paper we study another proof of Mochizuki's Galois theory without using p-adic Hodge theory and its generalization. This theory is applicable to a Diophantine problem and algebraic geometry from the point of view of birational geometry.

1 Galois Theory

Definition 1.1. *An extension L of a field k is said to be primary if the largest algebraic separable extension of k in L coincides with k .*

Proposition 1.2. *Let X be a k -scheme. The following statements are equivalent.*

- (a) *For every extension K/k , $X \otimes_k K$ is irreducible, i.e., geometrically irreducible.*
- (b) *For every finite separable extension K/k , $X \otimes_k K$ is irreducible.*
- (c) *X is irreducible and if x is a generic point, $k(x)$ is a primary extension of k .*

Proposition 1.3. *Let Ω be an algebraically closed field of K and all extensions of K subextensions of Ω . N a Galois extension of a field K , E any extension of K and $L = N \cap E$. Then the fields E and N are linearly disjoint over L , i.e., $E(N) \cong E \otimes_L N$.*

$$\mathrm{Gal}(E(N)/E) \cong \mathrm{Gal}(N/(E \cap N))$$

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Table 1:

$$\begin{array}{ccc}
N & \rightarrow & E(N) \\
\uparrow & & \uparrow \\
L & \rightarrow & E \\
\uparrow & & \\
K & &
\end{array}$$

Corollary 1.4. *For a field F such that $E \subset F \subset E(N)$, one obtains*

$$F = E(F \cap N)$$

Corollary 1.5. *Let E_1 and E_2 be two Galois extensions of K such that $E_1 \cap E_2 = K$. Then E_1 and E_2 are linearly disjoint over K and $K(E_1 \cup E_2)$ is a Galois extension of K .*

$$\text{Gal}(K(E_1 \cup E_2)/K) \cong \text{Gal}(E_1/K) \times \text{Gal}(E_2/K) \quad (1)$$

Table 2:

$$\begin{array}{ccc}
E_1 & \rightarrow & K(E_1 \cup E_2) \\
\uparrow & & \uparrow \\
E_1 \cap E_2 & \rightarrow & E_2
\end{array}$$

Proposition 1.6. *Let E_1 and E_2 be two extensions of K such that $E_1 \cap E_2 = K$ and linearly disjoint over K . Then one obtains*

$$\text{Gal}(\overline{K(E_1 \cup E_2)}/K(E_1 \cup E_2)) \cong \text{Gal}(\overline{E_1}/E_1) \times \text{Gal}(\overline{E_2}/E_2)$$

Theorem 1.7 (Galois Correspondence). *Let L be an infinite Galois extension of F , $G = \text{Gal}(L/F)$. For every closed subgroup H of G , let L^H denote the fixed field of H . The correspondence*

$$K \mapsto \text{Gal}(L/K)$$

defined for all intermediate field extensions $F \subset K \subset L$ is an inclusion reversing bijection between the set of all intermediate extensions K and the set of all closed subgroups of G . Its inverse is the correspondence

$$H \mapsto L^H$$

Table 3:

$$\begin{array}{ccccc}
\overline{E_1} & \rightarrow & \overline{E_1}(E_2) & \rightarrow & K(\overline{E_1} \cup \overline{E_2}) \\
\uparrow & & \uparrow & & \uparrow \\
E_1 & \rightarrow & K(E_1 \cup E_2) & \rightarrow & \overline{E_2}(E_1) \\
\uparrow & & \uparrow & & \uparrow \\
E_1 \cap E_2 & \rightarrow & E_2 & \rightarrow & \overline{E_2}
\end{array}$$

defined for all closed subgroups H of G . The extension K/F is normal if and only if $\text{Gal}(L/K)$ is a normal subgroup of G and one obtains the exact sequence

$$1 \rightarrow \text{Gal}(L/K) \rightarrow \text{Gal}(L/F) \rightarrow \text{Gal}(K/F) \rightarrow 1$$

Theorem 1.8. Let K be a field of characteristic 0, n, m positive integers and let x_1, \dots, x_n be transcendental indeterminates over K .

Table 4:

$$\begin{array}{ccc}
\overline{K(x_1, \dots, x_n)} & & \\
\uparrow & \nwarrow & \\
K(x_0, x_1, \dots, x_n) & \longrightarrow & \overline{K(x_1, \dots, x_m)}(x_{m+1}, \dots, x_n) \\
\uparrow & & \uparrow \\
K(x_0, x_1, \dots, x_n) \cap \overline{K(x_1, \dots, x_m)} & \longrightarrow & \overline{K(x_1, \dots, x_m)} \\
\uparrow & \nearrow & \\
K(y_0, x_1, \dots, x_m) & &
\end{array}$$

Then one has

$$\text{Gal} \left(\overline{K(x_1, \dots, x_m)}(x_{m+1}, \dots, x_n) / K(x_0, x_1, \dots, x_n) \right) \cong$$

$$\text{Gal} \left(\overline{K(x_1, \dots, x_m)} / K(x_0, x_1, \dots, x_n) \cap \overline{K(x_1, \dots, x_m)} \right)$$

Conversely, one can construct x_0 and y_0 in $\overline{K(x_1, \dots, x_n)}$ and $\overline{K(x_1, \dots, x_m)}$, respectively. Profinite groups have cohomological dimensions and let cd denote the cohomological dimension. Then $\text{cdKer}(\text{Gal}(/K(x_1, \dots, x_n)) \rightarrow \text{Gal}(/K))$ coincides with the transcendental dimension of an extension $\overline{K(x_1, \dots, x_n)}/\overline{K}$.

Table 5:

$$\begin{array}{ccc}
\mathrm{Gal}\left(\overline{K(x_1, \dots, x_n)}/K(x_0, x_1, \dots, x_n)\right) & \rightarrow & \mathrm{Gal}\left(\overline{K(x_1, \dots, x_m)}/K(x_0, x_1, \dots, x_n) \cap \overline{K(x_1, \dots, x_m)}\right) \\
& \searrow & \downarrow \\
& & \mathrm{Gal}\left(\overline{K(x_1, \dots, x_m)}/K(y_0, x_1, \dots, x_m)\right)
\end{array}$$

$$\mathrm{Hom}(\mathrm{Spec}(K(x_0, x_1, \dots, x_n)), \mathrm{Spec}(K(y_0, x_1, \dots, x_m)))$$

$$\cong \mathrm{Hom}_{\mathrm{Gal}(\overline{K}/K)}^{\mathrm{opencont}}(\mathrm{Gal}\left(\overline{K(x_1, \dots, x_n)}/K(x_0, x_1, \dots, x_n)\right), \mathrm{Gal}\left(\overline{K(x_1, \dots, x_m)}/K(y_0, x_1, \dots, x_m)\right))$$

Theorem 1.9. *Let K be an algebraic number field and X a complete normal variety of general type over K . $X(K)$ is not dense in X .*

Proof. Assume the conclusion is not true. One will prove the theorem by absurdity. Let $U_\beta \subset U_\alpha$ be open subvarieties of X for a partial order $\alpha \geq \beta$.

Table 6:

$$\begin{array}{ccccccc}
1 & \rightarrow & \pi_1(\overline{U_\alpha}) & \rightarrow & \pi_1(U_\alpha) & \rightarrow & \pi_1(K, \overline{K}) \rightarrow 1 \\
& & \uparrow & & \uparrow & & \uparrow \\
1 & \rightarrow & \pi_1(\overline{U_\beta}) & \rightarrow & \pi_1(U_\beta) & \rightarrow & \pi_1(K, \overline{K}) \rightarrow 1
\end{array}$$

Here the vertical upward arrows are surjective homomorphisms. The proof is continued in the later page. \square

Let π be a profinite group. Let $C(\pi)$ be a category of the finite sets on which π acts continuously.

Lemma 1.10 (SGA1). *The category $\mathrm{Pro}-C(\pi)$ of the pro-objects of $C(\pi)$ is canonically equivalent to the category $C'(\pi)$ of the profinite spaces on which π acts continuously.*

Let S be a locally noetherian connected scheme. Let $a : \Omega \rightarrow S$ be a geometric point of S , where Ω is an algebraically closed field. Let C be a category of etale coverings of S . Let F be a functor from C to the category of the sets. For an etale covering X/S , $F(X)$ is the set of geometric points over a .

Lemma 1.11 (SGA1). *Let \mathcal{C} be a Galois category. The fundamental pro-objects are isomorphic, the fundamental functors are isomorphic.*

Lemma 1.12. *Canonical homomorphisms $\text{Aut}(U_\alpha) \rightarrow \text{Out}(\pi_1(U_\alpha))$ are epimorphisms with splitting.*

Proof. There is an identification of universal coverings of U_α . □

Continuity of proof. One has $\text{Out}(\varprojlim \pi_1(\overline{U}_\alpha)) = \text{Out}(\text{Gal}(\overline{R(\overline{X})}/R(\overline{X}))) = \text{Bir}(\overline{X})$.

Since $\text{Out}(\pi_1(\overline{U}_\alpha))$ is identified as a subgroup of $\text{Bir}(\overline{X})$, there are only finitely many such groups.

Consider

$$H^1(\pi_1(K, \overline{K}), \text{Out}(\pi_1(\overline{U}_\alpha))) \quad (2)$$

Replacing K by a finite extension F of K , one obtains

$$H^1(\pi_1(F, \overline{K}), \text{Out}(\pi_1(\overline{U}_\alpha))) = 1 \quad (3)$$

One can find an extension F/K such that for all

$$H^1(\pi_1(F, \overline{K}), \text{Out}(\pi_1(\overline{U}_\alpha))) = 1. \quad (4)$$

Hence $\pi_1(U_\beta) = \pi_1(\overline{U}_\beta) \times \pi_1(F, \overline{K})$ for all β . Therefore, one has

$$\text{Gal}(\overline{R(X)}/R(X)) = \text{Gal}(\overline{R(\overline{X})}/R(\overline{X})) \times \text{Gal}(\overline{F}/F).$$

It is absurd. □

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