

# The Basic Metaphor of Infinity and the Concept of a Point

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**Abstract:** What is the link between natural human cognition and sophisticated mathematical concepts? G. Lakoff introduced the Basic Metaphor of Infinity, and other important metaphorical concepts and mappings to explain the gap. In this article, I would like to shed light on the concept of point of size zero in geometry. Geometrically speaking, a point is a location without size. I want to point out that this is an idealized concept, and in the real thinking process we have another concept of a point which is as small as it can be but still has some inner structure. Without such a modified concept of a point, the naturally continuous motion cannot exist.

**Keywords:** cognitive science, mathematical idea analysis, the basic metaphor of infinity, natural continuity

## 1 Infinity as a wonder

From the ancient times, human beings have been interested in the mystical, magical, wondrous behavior of the infinity. Names of large places in the decimal system were given in the old Indian Buddhist literature, which were transmitted beyond millennia to today's number systems in China and Japan.

Ancient Indian mathematicians tried to include the infinite and infinitesimal into the number system and wrote down the calculation table for them. Ancient Greek mathematicians like Archimedes treated some special cases of definite integrals in their theses, trickily avoiding infinity itself, but rather paraphrasing the problem to an argument about approximate values of an unknown area. What those scholars invented was a method of argument which could be repeated again and again without limitations to obtain an absolute consequence through the use of what we call today *the totally ordered set of the real number system*.

In the 17th century, Newton and Leibniz independently established a fairly general explanation of the reason for the integral to be calculated through an “inverse operation” of differentiation. It should be noted here that Leibniz not only introduced the notation

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Received Sept. 24, 2005

$dx$ , but also introduced the notion carried by the symbol  $dx = \lim_{\Delta x \rightarrow 0} \Delta x$  as an infinitesimal quantity.

In the 18th century, Cauchy and Weierstrass developed a new tool for formalizing limits. In their viewpoint, infinity lies in the infinite process where a number  $s_n$  approaches another number  $s$ , the limit. Each number was represented as a point on a line, called the *real line*, with the limit value corresponding to the limit point. The process was defined to be “convergent” if it satisfies the condition that the values of the number  $s_n$  approach arbitrarily closely a number  $s$  for  $n$  large enough. Here, “arbitrarily closely” means “closer than any positive number  $\varepsilon$ ”, i.e.,  $|s_n - s| < \varepsilon$ , and “for  $n$  large enough” means that there must be an  $N$  such that the above estimate is true for all  $n \geq N$ . By using these symbols, the infinite process of approaching a limit could be defined statically. With the use of quantifiers “any” and “exist”, calculus became arithmetic in the first order logic. Now there was neither movement nor naturally continuous lines. Calculus thus formalized became one branch of formal logic, which was certain, static, rigorous, but which prohibited movement from mathematics.

It is worth mentioning that, at this point, mathematics became a methodology to describe, reason, and prove mathematical truth concerning what is continuously moving in its infinite steps, using finite, discrete arrays of symbols. Separation of the sophisticated, technical mathematics from the human mathematics was clear. Human mathematics stayed alive in schools and in the folklore, whereas the technical mathematics became more and more sophisticated, rigorous, stable, static, discretized, and superficially literal.

## 2 A new metaphor theory

A metaphor is originally thought of as a method of expression in literature, especially in poems. According to OALD \*<sup>2</sup>, metaphor is “a word or phrase used in an imaginative way to describe sb/sth else, in order to show that the two things have the same qualities and to make the description more powerful, for example, *She has a heart of stone*.”

The new metaphor theory started with Lakoff & Johnson (1980). They pointed out from the standpoint of cognitive psychology, that metaphor is an indispensable tool for human language, cognition and behavior. In this context, language is the key natural phenomenon that plays the major role in human activities. Many linguistic studies show that there are some patterns of metaphorical connection between concepts which are

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\*<sup>2</sup> Oxford Advanced Learner's Dictionary of Current English.

common to many languages over the world.

Why is metaphor so important in human cognition? Language is not a mere tool for communication. We perceive the world around us, infer conclusion from a known fact, control the movement of our body, inspect our own feelings, and we do all these things using our language. Whenever the brain is working to perceive, think and control, it is using the language unconsciously and automatically. Language is a model for the world we cope with, and as such it must be as coherent and systematic as it can be. But, in reality, we ceaselessly discover new things and new phenomena, and generally we do not have the appropriate word for the new situation. It is thus necessary to use old language in a new situation. Without the mechanism for metaphorical concepts, we could not make progress at all.

### 3 The Basic Metaphor of Infinity

In this section, we discuss the Basic Metaphor of Infinity, or the BMI for short, as proposed in [8].

#### 3.1 Child's acquisition of the natural numbers

N. Chomsky once said that the concept of natural numbers —the concept of an infinite linear sequence of one thing after another —is the basis of human ability of language. For him, there is no distinct boundary between the biological bases of the human brain to process language and that of the ability to do mathematics. Indeed, the language ability is universal for the human species, and so is the ability to create, understand and transmit mathematical ideas.

Also, anatomically speaking, there is no part of the brain that serves exclusively for the human mathematical activities. When we measure the activation level of the parts of the brain of the testee solving a mathematical problem, we observe that various parts of the brain, especially those parts near the speech center, are activated simultaneously.

Acquisition of the concept of natural numbers seems to be fundamental for human development. I once observed how infants get the concept of whole numbers using my own son. First, he learned the names of natural numbers from one and two up to ten, but at that stage he didn't seem to understand the quantitative facet of them. He just mimicked the magical words in a queue, stimulated one by one by a picture card suggesting the number. Then, one day he found five bottles of the same size, brought them and put

them on tatami in a line, and happily he started to count them aloud. That was the moment he had a flash of inspiration what the numbers are for.

From this experiment, it seems that the natural numbers as a sequence of (meaningless) words or symbols come first, then sudden instinct opens for the child the world of quantity. Or, maybe the concept of quantity is not firmly established at this stage. Infants just gather various things and enjoy counting, without knowing what counting means. Then, only after conducting experiments sufficient number of times, they will discover the quantitative facets of numbers like the commutativity law  $a + b = b + a$ , and so on.

In this context, my view of human number development agrees with Kronecker's thought [9]. The essential property of the natural numbers is that it is one typical (the smallest) well-ordered set. It is a sequence of symbols where each symbol is followed by one and only one other symbol. Kronecker pointed out that all other arithmetical properties of the natural numbers follow logically from the well-ordered-ness.

All human beings, regardless of culture or education, can instantly tell at a glance whether there are one, two, or three objects before them. This ability is called *subitizing*. According to recent psychological studies, a baby, at three or four days, can discriminate between collections of two and three items (Antell & Keating, 1983). This showed that subitizing is an innate ability.

The ability of subitizing is restricted up to three or four items in sight. For sounds or flush of light arranged along some time length, the range gets a little bit wider, but anyway there is a limitation to this innate ability. Subitizing occurs instantly without giving focus on each of the items one by one, and it occurs before putting attention on the items. It seems that subitizing uses a totally different mechanism from counting. When the number of items exceeds the range of subitizing (i.e. about four), people begin to count, which is totally different procedure from instinctive comprehension.

### 3.2 How is infinity embodied?

I think Kronecker and Chomsky, though one is a mathematician and the other is a linguist, pointed out one common fact about human cognition. Human beings have the ability to recognize the infinity, the ability to understand the situation where some kind of things (symbols or words or anything) stand in line in space or in time in such a way that every item in the line is followed by another item.

The most essential property of this situation is that it has no end. To begin to see the embodied source of the idea of infinity, we must look to one of the most common of human

conceptual systems. Linguists call it the *aspectual system*. All the human languages in the world, without counterexamples, have this system. The aspectual system characterizes the structure of event-concepts—events as we conceptualize them. We experience that some actions are inherently iterative, whereas others are inherently continuous. In English, ‘tap’, ‘breathe’, and ‘move’ are typical verbs that have imperfective aspect. This means that they do not have completions. Of course, in life, hardly anything one does goes on forever. What matters is how we conceptualize them. If we stop breathing, it is not the completion but the interruption of breathing. We cannot stop breathing and say, for example, “I have breathed enough”. The act of breathing can stop, but we never conceptualize it as the completion of the act. The concept of aspect appears to be embodied in the motor-control system of the brain.

It seems that the aspectual system in our brain is the fundamental source of the concept of infinity.

### 3.3 Potential infinity and actual infinity

Ongoing process or motions without end was called *potential infinity* by Aristotle. He distinguished it from *actual infinity*, which is infinity conceptualized as a realized “thing”. Potential infinity shows up in mathematics all the time, but the interesting cases on infinity in modern mathematics are cases of actual infinity.

It seems that the idea of actual infinity in mathematics is metaphorical in nature. There seems to be one common mechanism which produces the ultimate metaphorical *result* of a process without end. This conceptual metaphor is called the Basic Metaphor of Infinity. By this metaphor, processes that go on indefinitely are conceptualized as having an end and an ultimate result. The effect of BMI is to add a new element—a metaphorical completion—to the ongoing process.

## 4 Some case studies

### 4.1 Children’s concept of points

In the 1940s, child psychologists Jean Piaget and Bärbel Inhelder established the following experimental results for children around the age of seven or younger. When you tell a child to imagine a disc (or a dot or circle) and make it smaller and smaller and smaller until it’s as small as it can get, the child will conceptualize something that is as small as it can be but is still a bona fide disc, with a center separate from the circumference (Piaget

& Inhelder, 1948). Suppose a child is told: Start with one of various figures—a circle, triangle, or square—and make it as small as it can get. What the child gets is a point. Now make the point larger and larger. What does it look like? The answer is the same figure the child started with—the circle, triangle, or square. The figures do not lose their integrity and collapse to the mathematical point. Moreover, when children were asked to shrink an object with volume—say, a ball—to a point, they said that points had volume (see Núñez, 1993).

In order to teach mathematics, one must teach the difference between everyday concepts and technical concepts. In general, technical concepts have metaphorical nature compared with everyday concepts. In the above context, children's concept of "point" is a disc (or a dot or circle) with its diameter as small as it can be. Children's concept of a point is different from the geometric point Euclid stated as "something that has no part". According to Euclid's technical definition, a point cannot have a center separate from its circumference.

Natural conceptualization of a point in everyday life is, mathematically speaking, more like an infinitesimally small disc. Among the planar figures we encounter in everyday life, a disk or circle is the only figure that is symmetric in every direction. And a point must be symmetric in every direction. But the second experiment above shows that children's concept of point can also have other types of figures. It can be a circle, triangle, or square, with its size infinitesimally small.

## 4.2 Exceptional points in continuous motion

Suppose that a point  $P$  moves along the circle of diameter  $AB$ . Then  $\triangle ABP$  is a right-angled triangle with its hypotenuse  $AB$ . Draw the square with edge  $PA$  (resp.  $PB$ ) on that triangle touching from outside. Then, according to Pythagorean theorem, the total area of the two squares remains the same all through the motion of the vertex  $P$  along the arc between the points  $A$  and  $B$  (see Fig ??).

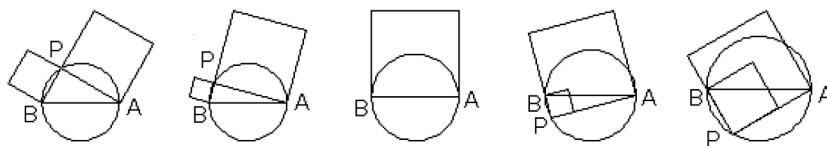


Fig 1: Pythagorean theorem

What happens when  $P$  moves beyond the point  $B$ ? When  $P$  goes by the point  $B$ , the

distance between P and B will become smaller and smaller. At the very moment P passes through B, this distance becomes zero for only an instance, then the next moment, the point P appears on the other side of the point B, with the segment PB flipped over. All the other edges of this square becomes a point (the same point as B) for an instant, then the total figure will appear again with a finite size, but this time it is turned upside down and the right side left. As a consequence, the smaller square will be included in the larger one. Although Pythagorean theorem itself holds continuously, the two squares begin to overlap with the triangle for a while.

It seems to us that all the points defining this figure are moving continuously and smoothly with the total area of the two squares staying constant, but there is one moment when the four vertices coincide with each other and the left square collapses into a point of area zero.

This is an exceptional moment in the movement, where segments connecting vertices of the square disappear for an instant, but we feel something kept continuous. For example, the lines going through any pair of two vertices of the square move continuously. When segments disappear, one of the lines will become the tangent line to the circle: it will be tangent to the locus of the point.

Such a circumstance is common in mathematics. Tangent lines are human creation that fit to exceptional cases. Tangents and derivatives are as metaphorical ideas as infinity. Just as a telescope held in hand upside-down will serve as a microscope, so the Basic Metaphor of Infinity can be used to create infinitesimal objects.

## 5 Conclusion

Metaphorical reasoning is pervasive in our mathematical activities as well as everyday activities. We use metaphors both in creating and learning new mathematical concepts. The layers of metaphors in mathematics are very complex. The metaphorical study of mathematics has just started, and most of the complicated metaphorical network structure of the brain is yet to be discovered. Anyway, we must first admit the importance of the metaphor in mathematics, although in some cases it may deceive us. Then we can go further.

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