Comparison Theory

Kazuhisa Maehara*

Abstract

In this article we shall propose a new formula which is a generalization of Iitaka's addition formula and we shall prove it. It works for varieties of negative Kodaira dimension.

1 Introduction:

To classify algebraic varieties in the category of birational geometry, Iitaka proposed many conjectures after Kodaira-Enriques classification of surfaces. His key birational invariant is Kodaira dimension. One of his main conjectures is the following:

Conjecture 1.1. *Let* $f : X \to S$ *be a fibre space over a field of characteristic* 0*. Then* $\kappa(X) \geq \kappa(X_s) + \kappa(S)$*, where* X_s *is a general fibre of* X/S*.*

Vanishing theorems, weak positivity and extremal rays are found to prove his conjectures. Viehweg conjectures the following:

Conjecture 1.2. *Let* X/S *be a fibre space with the geometric generic fibre of Kodaira dimension* ≥ 0*. Then there exists a number* m *such that*

$$\kappa(\det f_*\omega_{X/S}^{\otimes m}) \geq \mathrm{var}(X/S).$$

This is equivalent to the following

Conjecture 1.3. $\kappa(\omega_{X/S}) \geq \kappa(\omega_{X_s}) + \mathrm{var}(X/S)$

This implies Iitaka conjecture.
It shall be derived from the following:

Conjecture 1.4. *Assume* $\mathrm{var}(X/S) = \dim S$ *and* $\kappa(X_s) \geq 0$*. Then* $f_*\omega_{X/S}^{\otimes m}$ *is big for some* $m > 0$*.*

---

*1 Associate Professor, General Education and Research Center, Tokyo Polytechnic University
Received Sept. 8, 2005

Viehweg conjecture seems to be the strongest one among analogs of Iitaka conjecture. These conjectures work well in the category of varieties of Kodaira dimension ≥ 0. In this category rational maps have functorial properties and the connected component of the birational automorphism group of varieties of Kodaira dimension ≥ 0 is algebraic. The varieties of Kodaira dimension $-\infty$ is troublesome for functorial property. The varieties are to be uniruled. It would have Mori fibre space. Even though the structure of varieties of Kodaira dimension $-\infty$ is rather simple, the rational maps between them are occasionally uncontrollable. The connected component containing "id" of birational automorphism groups for almost all varieties are non algebraic.

Conjecture 1.5. *The canonical invertible sheaf is weakly positive if and only if the variety is of Kodaira dimension* ≥ 0.

Iitaka furthermore has proposed a program for varieties with logarithmic structure to classify varieties of Kodaira dimension $-\infty$.

Unexpectedly the author found the following theorem which works in the category of varieties of Kodaira dimension $-\infty$.

Theorem 1.6. *Let* $f : X \to S$ *be a fibre space with* $\mathrm{var}(X/S) = \dim S$ *and* ω_X *weakly positive. Then there exists a big* $\mathbb{Q}$ *invertible sheaf* L *over* S *such that* $\kappa(f^*L^{-1} \otimes \omega_{X/S}) \geq 0$ *and* $\kappa(L \otimes \omega_S) \geq 0$.

This theorem implies almost all conjectures above.

For completing inductive arguments, let's introduce a category such that its object is a couple (X, A_X) of a projective variety X with terminal singularities and a big $\mathbb{Q}$ invertible sheaf or the structure sheaf $\mathcal{O}_X$ over X described by A_X, for simplicity $(X, A) = (X, A_X)$, its morphism is an ordinary morphism and let $\omega_{(X,A)}$ denote $\omega_X \otimes A_X$.

Theorem 1.7. *Let* $f : X \to S$ *be a fibre space with* $\omega_{(X,A)}$ *weakly positive and* $\mathrm{var}(X/S) = \dim S$. *Then there exists a big* $\mathbb{Q}$ *invertible sheaf* L *over* S *such that* $\kappa(\otimes\omega_{(X,A)/(S,L)}) \geq 0$ *and* $\kappa(\omega_{(S,L)}) \geq 0$.

As to a variety with $\omega_{(X,A)}$ not weakly positive, one obtains

Theorem 1.8 (Rationality)**.** *Assume that a canonical divisor* K_X *of* X *is not weakly positive and that* D *is big. Then* $t_0 = \inf\{0 < t \in \mathbb{Q}; K_X + tD \textit{is weakly positive}\}$ *is a rational number. Furthermore,* $\dim X > \kappa(K_X + t_0 D) \geq 0$.

2 Proofs

2.1 preliminaries

First one investigates the birational geometry from the point of view of the absolute Galois groups. Let $X \to S$ be a fibre space. Let $1 \to G \to E \to K \to 1$ be an extension of a profinite group K by a profinite group G associated to the fibre space $X \to S$. To an exact sequence $1 \to \mathrm{InnG} \to \mathrm{Aut}G \to \mathrm{Out}G \to 1$, we have an exact sequence

$$\mathrm{H}^1(\mathrm{B}K, \mathrm{Inn}G) \to \mathrm{H}^1(\mathrm{B}K, \mathrm{Aut}G) \to \mathrm{H}^1(\mathrm{B}K, \mathrm{Out}G),$$

i.e.,

$$\mathrm{Hom}(K, \mathrm{Inn}G) \to \mathrm{Hom}(K, \mathrm{Aut}G) \to \mathrm{Hom}(K, \mathrm{Out}G).$$

Here $\mathrm{B}K$ denotes the classifying space of K and $\mathrm{Out}G$ denotes the outer automorphism group of G. A group extension is an element of $\mathrm{H}^1(\mathrm{B}K, G \to \mathrm{Aut}G)$, where $G \to \mathrm{Aut}G$ is a crossed module. We have

$$1 \to \mathrm{H}^2(\mathrm{B}K, \mathrm{Z}(G)) \to \mathrm{H}^1(\mathrm{B}K, G \to \mathrm{Aut}G) \to \mathrm{H}^1(\mathrm{B}K, \mathrm{Out}G).$$

Here $\mathrm{Z}(G)$ denotes the center of G. Assume that a general fibre of X/S has non negative Kodaira dimension. Then the canonical representation $\rho : K \to \mathrm{Out}G$ turns out to be trivial after replacing a profinite group associated to a finite morphism $S' \to S$ in the following lemma. Furthermore assume that the extension is neutral.

Since we have $\mathrm{H}^2(BK, \mathrm{Z}(G)) \to \mathrm{H}^1(BK, G \to \mathrm{Aut}(G))$, the extension $1 \to G \to E \to K \to 1$ is given by pushing out an extension $1 \to \mathrm{Z}(G) \to E' \to K \to 1$. Hence E' is a semi-direct product $\mathrm{Z}(G) \rtimes K$, which is contained in a semi-direct product $G \rtimes K$. Thus this central extension is trivial. Therefore by pushing out this central extension, the extension $1 \to G \to E \to K \to 1$ is trivial.

Lemma 2.1. *Assume the geometric generic fibre of X/S has non negative Kodaira dimension. There exists a generically finite morphism $S' \to S$ such that the representation $\rho : K' \to \mathrm{Out}(G)$ is trivial. Here K' denotes the absolute Galois group* $\mathrm{Gal}(\overline{R(S')}/R(S'))$.

Proof. Let A denote $\mathrm{Out}(G)$. Since the generic fibre of X/S is not uni-ruled, the birational automorphism group A of the generic fibre is locally algebraic. The natural representation $\rho : K \to A$ induces $\overline{\rho} : K \to A/A^0$, where A^0 denotes the neutral component of A. There is no countable profinite group. Since A/A^0 is a countable set, $\overline{\rho}(K)$ is a finite group. Replace by K the kernel of $\overline{\rho}$. We have $\rho : K \to A^0$. Hence we have an isomorphism

$$H^1(\overline{R(S)}/R(S), A^0(\overline{R(S)}) \cong H^1(BK, A^0).$$

Let P be an A^0-torsor associated to $\rho : K \to A^0$. A^0 is algebraic (quasi-compact, faithfully flat and of finite type) over Spec(R(S)). Thus there exists a generically finite $S' \to S$ such that an A^0-torsor P is trivial over Spec(R(S')). Hence the representation $\rho : K' \to \mathrm{Out}(G)$ is trivial. □

We obtain the following key result in our proof.

Theorem 2.2. *Let* $1 \to G \to E \to K \to 1$ *be an extension of a profinite group* K *by a profinite group* G*. Assume*

(a) the connected component containing id, i.e., the neutral component of $\mathrm{Out}(G)$*, is an algebraic group.*

(b) $E \to K$ *has a section which is a group homomorphism, i.e., a neutral extention.*

Then there exists a profinite group K' *such that the pull-back of the extension* $1 \to G \times_K K' \to E \times_K K' \to K' \to 1$ *is a direct product.*

By Galois theory and Mochizuki's theorem([Mch]). we have a categorical equivalence between the category of complete varieties as objects with dominant rational maps as morphisms and that of bands (liens in French) of profinite groups.

Note that the neutral component of the birational automorphism group $Bir(X)$ of a complete variety of non negative Kodaira dimension is a smooth algebraic group and that the extension of the rational function fields of a fibre space X/S is a regular extension.

To a fibre space X/S corresponds an extension E of K by G. One can interpret the theorem above.

Remark 2.3. *Let* $f : X \to Y$ *be a dominant rational map over* S*. Assume that* $Bir(Y)$ *is locally algebraic. Then one obtains* $\mathrm{var}(X/S) \geq \mathrm{var}(Y/S)$*.*

Remark 2.4. *Let* V *be a quartic uniruled threefold. Then any deformation of a quartic threefold* V *over a curve with maximal variation contains only discrete quartic threefolds.*

Remark 2.5. *Let* Γ *be the absolute Galois group of the rational function field of the projecive line over an algebraically closed field of characteristic* 0*.* Γ *has no center and* $Out(\Gamma)$ *an algebraic group. Hence the absolute Galois groups of the rational function fields of rational varieties are center-free. Consider a fibre space with a general fibre rational. There exists a profinite group* K *such that* $H^1(K, \Gamma \to Out(\Gamma)) = 1$*. Furthermore,* $H^1(K, \Gamma_u \to Out(\Gamma_u)) \cong H^1(K, Out(\Gamma_u))$*.*

Question 1: Are the absolute Galois groups of the rational function fields of unirational varieties (resp. rationally connected varieties) center-free?

Remark 2.6. *As to an extension $1 \to G \to E \to K \to 1$, if G and K are center-free, then E is center-free. The image of $Z(E)$ is contained in $Z(K)$.*

Quetion 2: If both G and K have non trivial centers, when has E a non trivial center larger than $Z(G)$ under some conditions ?

Question 3: Have the absolute Galois groups of the rational function fields of non uniruled varieties non trivial centers?

Question 4: What conditions do the absolute Galois groups of the rational function fields of non uniruled varieties (resp. varieties of Kodaira dimension non negative) satisfy ? Will they be the conditions (i)the neutral component of the outerautomorphism groups of the absolute Galois groups are algebraic, (ii) the centers of the absolute Galois groups are non trivial ?

Remark 2.7. *The absolute Galois groups of the rational function fields of varieties are successive extensions of those of the rational function fields of curves.*

The absolute Galois groups of the rational function fields of non uniruled varieties have centers. The absolute Galois groups of the rational function fields of varieties have quotient groups which are non abelian finite groups.

Remark 2.8. *Simple profinite groups are not associated to the absolute Galois groups of rational function fields.*

Remark 2.9. *Let $\phi : G_1 \to G_2$ be an open continuous homomorphism of profinite groups. $\phi(G_1) \subset G_2$. Let $Z(G_2)C_{\phi(G_1)}(\phi(G_1))$ denote C. Then for a homomorphism between extensions of K by G_1 and G_2 respectively, one has homomorphisms $H^2(K, Z(G_1)) \to H^2(K, \phi(Z(G_1))) \to H^2(K, C)$. There exists an open subgroup K' of finite index of K such that $H^2(K', Z(G_2)) \to H^2(K', C)$ is injective.*

Remark 2.10. *Consider a fibre space with the geometric generic fibre with the birational automorphism group locally algebraic. Let Γ be the absolute Galois group of the geometric generic fibre. There exists a profinite group K of finite index of the profinite group associated to the base variety such that $H^1(K, \Gamma \to Out(\Gamma)) \cong H^2(K, Z(\Gamma))$.*

Let X/S be a fibre space. Consider the following commutative diagram:

$$
\begin{array}{ccc}
X & & \\
 & g \searrow & \\
f \downarrow & & Z \\
 & \swarrow h & \\
S & &
\end{array}
$$

This formula is available.

Lemma 2.11 (Iitaka). *For a fibre space X/S and an invertible sheaf L, $\kappa(L) \leq \kappa(L|X_{\overline{\eta}})+\dim S$*

One has the following two lemmas for variations of fibre spaces.

Lemma 2.12.

$$\mathrm{var}(X/S) \leq \mathrm{var}(X/Z) + \mathrm{var}(Z/S)$$

Proof. One begins with the case $\mathrm{var}(Z/S) = 0$. Hence one assume that there exists a variety such that $Z/S = F \times S$ for simplicity. For any point $t \in F$ one has a fibre space $X \times_Z \ S \times t = X_t$. If $\mathrm{var}(X_t/S_t) < \dim S_t$, there exists a fibre space X_0/S_0 such that X_t/S_t is the pull-back of X_0/S_0 and $\dim S_0 = \mathrm{var}(X_t/S_t)$. Hence there exists a variety X^0 such that $X \sim X^0 \times_{S_0 \times F} S \times F$. Thus $\mathrm{var}(X/S) \leq \dim S_0$, which is a contradiction. In general there exists a fibre space Z_0/S_0 such that X/S is the pull-back of X_0/S_0. One restricts the diagram $Z/S \to Z_0/S_0$ over any point $t \in S_0$. Then $\mathrm{var}(Z_t/S_t) = 0$. Thus $\mathrm{var}(X_t/S_t) \leq \mathrm{var}(X_t/Z_t)+\mathrm{var}(Z_t/S_t)$ for any point t. If $\mathrm{var}(X/S) > \mathrm{var}(X/Z)+\mathrm{var}(Z/S)$, then there exists a point such that $\mathrm{var}(X_t/S_t) > \mathrm{var}(X_t/Z_t) + \mathrm{var}(Z_t/S_t)$, which is a contradiction. □

Lemma 2.13. *Let Z/S the pull-back of Z_0/S_0. For the geometric generic poit $\overline{\eta}_0$ of S_0 let $X_{\overline{\eta}_0}/Z_{\overline{\eta}_0}/S_{\overline{\eta}_0}$ be the pull-back $X/Z/S$. Then $\dim S_{\overline{\eta}_0} \leq \mathrm{var}(X_{\overline{\eta}_0}/Z_{\overline{\eta}_0})$*

Proof. Note that $\mathrm{var}(X_{\overline{\eta}_0}/S_{\overline{\eta}_0}) \leq \mathrm{var}(X_{\overline{\eta}_0}/Z_{\overline{\eta}_0}) + \mathrm{var}(Z_{\overline{\eta}_0}/S_{\overline{\eta}_0})$ and $\mathrm{var}(Z_{\overline{\eta}_0}/S_{\overline{\eta}_0}) = 0$. Since $\dim S_{\overline{\eta}} = \mathrm{var}(X_{\overline{\eta}}/S_{\overline{\eta}_0})$, one has $\dim S_{\overline{\eta}} \leq \mathrm{var}(X_{\overline{\eta}_0}/Z_{\overline{\eta}_0})$. □

2.2 Weak positivity

Let S be a projective smooth variety over a field of characteristic 0 and $\mathcal{G}$ a coherent sheaf over S.

Definition 2.14. *A coherent sheaf $\mathcal{G}$ is said to be weakly positive with respect to an invertible sheaf L over a dense open subset S^o of S if for every number $\alpha > 0$ there exists a number $\beta > 0$ such that the canonical homomorpfism*

$$\mathcal{O}_S \otimes H^0(S, S^{\alpha\beta}\mathcal{G} \otimes L^{\otimes\beta}) \longrightarrow S^{\alpha\beta}\mathcal{G} \otimes L^{\otimes\beta}$$

is surjective over S^o.

A coherent sheaf $\mathcal{G}$ is said to be big if for a big invertible sheaf H there exists a number $\nu > 0$ such that $S^{\nu}\mathcal{G} \otimes H^{-1}$ is weakly positive.

Let $f : X \to S$ be a fibre space between smooth projective varieties with $\kappa(\omega_{(X,A)}|\overline{\eta}) \geq 0$. Suppose there exists a number $m \geq 2$ such that $f^* f_* \omega^{\otimes m}_{(X,A)/S} \to \omega^{\otimes m}_{(X,A)/S}$ is generically surjective. Let H be an ample invertible sheaf over S. Put

$$r(\nu) = \mathrm{Min}\{\mu; f_*\omega^{\otimes \nu}_{(X,A)/S} \otimes H^{\otimes \mu\nu - 1} \text{is weakly positive}\}.$$

Then there exists a number $\beta > 0$ such that

$$S^\beta(f_*\omega^{\otimes \nu}_{(X,A)/S}) \otimes H^{\otimes \beta \cdot r(\nu)\cdot \nu - \beta} \otimes H^{\otimes \beta} = S^\beta(f_*(\omega^{\otimes \nu}_{(X,A)/S} \otimes f^* H^{\otimes r(\nu)\cdot \nu}))$$

is globally generated over some dense open subset. Put $r = r(m)$. Composing homomorphisms one has a globally generated invertible sheaf

$$\omega^{\otimes m}_{(X,A)/S} \otimes H^{\otimes r\cdot m}$$

over an inverse image of some open subvariety of S. One can represent this invertible sheaf by an effective divisor which has no fixed part on an open set where the invertible sheaf is globally generated. One chooses a general effective divisor such that a moving component has no multiplicity. Put $\mathcal{O}(D) = \omega^{\otimes m}_{(X,A)/S} \otimes H^{\otimes r \cdot m}$. Desingularize X such that all the fractional components are normally crossing and replace it by X. Consider a multi Kummer m-covering T with respect to all the fractional components which are nomally crossing. Then

$$f_*\omega_{(X,A)/S}(\lceil \omega^{\otimes (m-1)}_{(X,A)/S} \otimes H^{\otimes r\cdot (m-1)}(-\frac{m-1}{m}D)\rceil)$$

is a direct factor of $g_*\omega_{T/S}$. The sheaves

$$f_*\omega_{(X,A)/S}(\lceil \omega^{\otimes (m-1)}_{(X,A)/S} \otimes H^{\otimes r\cdot (m-1)}(-\frac{m-1}{m}D)\rceil)$$

and

$$f_*(\omega^{\otimes m}_{(X,A)/S} \otimes H^{\otimes r\cdot (m-1)})$$

have the same ranks. Assume $g_*\omega_{T/S}$ is weakly positive. Hence $f_*\omega^{\otimes m}_{(X,A)/S} \otimes H^{\otimes r\cdot(m-1)}$ is also weakly positive. By definition,

$$(r-1)m - 1 < r(m-1)$$

i.e., $r \leq m$. Hence $f_*\omega^{\otimes m}_{(X,A)/S} \otimes H^{\otimes (m^2-1)}$ is weakly positive. This number $\ell = m^2 - 1$ is independent of any ample invertible sheaf H and a pull-back of X/S to every finite smooth covering $\tau : S' \to S$. Give $\alpha > 0$. Take a Kawamata covering $\tau : S' \to S$ such that $\tau^* H = H'^{\otimes 2\alpha\ell + 1}$. Pull back a fibre space X/S to $X_{S'}/S'$ by the Kawamata covering. Apply the same argument to a desingularized fibre space $X_{S'}/S'$.

$$S^{\beta}(S^{2\alpha}(\tau^* f_* \omega^{\otimes m}_{(X,A)/S} \otimes H'^{\ell}) \otimes H')$$

is globally generated over a dense open subset. This is rewritten as

$$S^{\beta}(S^{2\alpha}(\tau^* f_* \omega^{\otimes m}_{(X,A)/S}) \otimes \tau^* H).$$

Thus

$$\mathcal{O}_{S'} \otimes H^0(S, S^{\beta}(S^{2\alpha}(\tau^* f_* \omega^{\otimes m}_{(X,A)/S}) \otimes \tau^* H)) \longrightarrow S^{\beta}(S^{2\alpha}(\tau^* f_* \omega^{\otimes m}_{(X,A)/S}) \otimes \tau^* H).$$

By the trace homomorphism $\tau_* \mathcal{O}_{S'} \to \mathcal{O}_S$, one obtains a composition of surjections

$$\tau_* \mathcal{O}_{S'} \otimes H^0(S, S^{\beta}(S^{2\alpha}(\tau^* f_* \omega^{\otimes m}_{(X,A)/S}) \otimes \tau^* H)) \longrightarrow S^{\beta}(S^{2\alpha}(f_* \omega^{\otimes m}_{(X,A)/S}) \otimes H \otimes \tau_* \mathcal{O}_{S'}) \longrightarrow$$

$$S^{\beta}(S^{2\alpha}(f_* \omega^{\otimes m}_{(X,A)/S}) \otimes H).$$

Thus tensoring $H^{\otimes \beta}$ one gets $S^{2\beta}(S^{\alpha}(f_* \omega^{\otimes m}_{(X,A)/S}) \otimes H)$ which is globally generated over a dense open set for a suitable $\beta >> 0$. Thus $f_* \omega^{\otimes m}_{(X,A)/S}$ is weakly positive. Therefore one can take the minimal number $r(m) = 1$ in the first assumption.

Proposition 2.15. *One has a generic surjection*

$$g_* \omega_{T/S} \to f_* \omega^{\otimes m}_{(X,A)/S} H^{\otimes (m-1)}.$$

This formula is independent of taking an ample invertible sheaf H.

Lemma 2.16. *Assume that $L^{-1} S^{\gamma} g_* \omega_{T/S}$ is weakly positive. Then*

$$L^{-1} S^{\gamma} f_* \omega^{\otimes m}_{(X,A)/S} \otimes H^{\otimes \gamma (m-1)}$$

is weakly positive, which is independent of taking an ample invertible sheaf H.

Proof. From proposition above

$$g_* \omega_{T/S} \to f_* \omega^{\otimes m}_{(X,A)/S} H^{\otimes (m-1)}$$

is generically surjective and so

$$L^{-1} \otimes S^{\gamma} g_* \omega_{T/S} \to L^{-1} \otimes S^{\gamma} f_* \omega^{\otimes m}_{(X,A)/S} H^{\otimes \gamma (m-1)}$$

□

Hence, one obtains the following

Proposition 2.17. *Assume that $g_* \omega_{T/S}$ is big. Then $f_* \omega^{\otimes m}_{(X,A)/S}$ is big.*

Remark 2.18. *In this case a canonical divisor of the geometric generic fibre of the multi m-Kummer covering T/S is abundant.*

Theorem 2.19. *Let X/S be a fibre space with a canonical invertible sheaf $\omega_{(X,A)}|\overline{\eta}$ of the geometric generic fibre abundant. Then there exists a number m such that $\kappa(\det\ f_*\omega_{(X,A)/S}^{\otimes m}) \geq$ $\mathrm{var}(X/S)$.*

One divides the proof into several steps.

First, one may reduce the theorem to the following. Take the Iitaka multi canonical fibring $\phi : X \to \mathbb{P}(f_*\omega_{(X,A)/S}^{\otimes m})$ over S.

$$\begin{array}{ccc} X & & \\ & \searrow & \\ f\downarrow & & Y \\ & \swarrow & \\ S & & \end{array}$$

Here Y is the desingularization of the image of ϕ and replace the other varieties by suitable desingularizations. By Iitaka's theorem the geometric generic fibre $(X, A)_y$ of $(X, A)/Y$ is of Kodaira dimension 0 and abundant.

Lemma 2.20.

$$\kappa(\det f_*\omega_{(X,A)/S}^{\otimes m}) \geq \mathrm{var}(Y/S)$$

Proof. It suffices to prove that if $\kappa(\det f_*\omega_{(X,A)/S}^{\otimes m}) = 0$, then $\mathrm{var}(Y/S) = 0$. But it is obvious. □

Assume $\dim Y/S > 0$. If $\mathrm{var}(Y/S) = \dim S$, there is nothing to be proved. If $\mathrm{var}(Y/S) = 0$, there exist so many sections that the closure of them is dense after replacing a generically finite covering $S' \to S$. Hence the inductive argument holds. There remains the case $\dim Y/S = 0$,i.e., $\kappa(\omega_{(X,A)}|s) = 0$. When the irregularity $q(X_s) = \dim H^1(X_s, \mathcal{O}_{X_s}) > 0$, take the relative Albanese mapping. The image over the base variety S has the geometric generic fibre of Kodaira dimension ≥ 0. Hence the inductive argument holds if $q(X_s) > 0$. Thus there remains the case $\kappa(\omega_{(X,A)}|s) = 0, q(X_s) = 0$ and $\omega_{(X,A)}|s$ is abundant.

Lemma 2.21. *Let X/S be a fibre space with a trivial canonical invertible sheaf $\omega_{(X,A)}$ of the geometric generic fibre and A a normal crossing divisor. Then there exists a number m such that $\kappa(\det\ f_*\omega_{(X,A)/S}) \geq \mathrm{var}(X/S)$.*

Proof. If the Griffiths period mapping ψ is generically finite, then $f_*\omega_{(X,A)/S}$ is big. Otherwise, there exists a subvariety S' such that the image ψ is a point. Replace S' by S. The tangential map of ψ $T_s \to Hom(H^0(X_s, \Omega^d_{X_s}\langle A\rangle), H^1(X_s, \Omega^{d-1}_{X_s}\langle A\rangle))$ is zero. Since $\omega_{(X,A)}|s = \mathcal{O}_{X_s}$, one obtains Kodaira-Spencer mapping $T_s \to H^1(X_s, \Theta_{X_s} \otimes \omega_{(X,A)}|s)$, which vanishes. Hence $\mathrm{var}(X/S) = 0$. This completes the proof. □

Lemma 2.22. *If $\omega_{(X,A)/(S,L)}$ is weakly positive and if A and L are invertible sheaves over X and S, respectively, then $f_*\omega_{(X,A)/(S,L)}$ is weakly positive, or zero sheaf. Furthermore, if $\omega_{(X,A)/(S,L)}$ is abundant over the geometric genric fibre of X/S, then there exists an integer $m > 0$ such that $f_*\omega^{\otimes m}_{(X,A)/(S,L)}$ is weakly positive.*

Proof. From Kollar-Viehweg's theorem([Vieh]), $f_*(\omega_{(X,A)}) \otimes H^{\otimes d+1}$is generated by the global sections when H is very ample over S and $d = \dim S$. Let $X^{(r)}$ denote a desingularization of r-power of $X^r = X \times_S \cdots \times_S X$. $f^{(r)}_*(\omega_{(X^{(r)},A)}) \otimes H^{\otimes d+1} = \omega_{(S,L)} \otimes f^{(r)}_*\omega_{(X^{(r)},A)/(S,L)} \otimes H^{d+1}$ is generated by the global sections. Since X^r is of Gorenstein singularity and X/S is able to be assumed flat outside a locus of codimension 2 of S, one has an isomrphism $f_*\omega_{(X,A)/(S,L)} = \otimes^r f_*\omega_{(X,A)/(S,L)}$. Hence $f_*\omega_{(X,A)/(S,L)}$ is weakly positive, or zero sheaf.

Let $f_1 : X_1 \to X$ be a multi-Kummer covering with respect to the fractional part of an effective $\mathbb{Q}$-ample divisor the support of which is normally crossing such that L is an invertible sheaf. Then $f_{1*}\omega_{(X_1,A)/(S,L)}$ is weakly positive. Since $\omega^{\otimes m}_{(X,A)/(S,L)}$ is an invertible sheaf over X and since $\omega_{(X,A)/(S,L)}$ is abundant over the geometric generic fibre of X/S, there exists an integer $m > 0$ such that $f_1^* f_{1*}\omega^{\otimes m}_{(X,A)/(S,L)} \to \omega^{\otimes m}_{(X,A)/(S,L)}$ is surjective. Let H be an ample invertible sheaf over S and

$$r(\nu) = \min\{\mu | f_{1*}\omega^{\otimes\nu}_{(X,A)/(S,L)} \text{is weakly positive}\}.$$

By the definition of weak positivity,

$$S^\beta(f_{1*}\omega^{\otimes\nu}_{(X,A)/(S,L)} \otimes H^{\otimes r(\nu)\nu-1} \otimes H) = S^\beta(f_{1*}\omega^{\otimes\nu}_{(X,A)/(S,L)} \otimes H^{\otimes r(\nu)\nu})$$

is spanned by the global sections over a dense open subset of S. Let $\nu = m$, $r = r(m)$. Thus, if necessary, replacing X_1 by a birationally equivalent variety, one can represent $\omega^{\otimes m}_{(X,A)/(S,L)} \otimes H^{\otimes m}$ as an effective divisor D such that the restriction of D on a general fibre of X_1/S is non singular and such that the support of the fractional part of D is normally crossing. Take again a multi-Kummer covering $X_2 \to X_1$ with respecr to the fractional part of $\frac{m-1}{m}D$. Then

$$f_{1*}\omega_{(X_1,A)/(S,L)}(\lceil\omega^{\otimes m-1}_{(X,A)/(S,L)} \otimes H^{\otimes r(m-1)}(-\frac{m-1}{m}D)\rceil)$$

is weakly positive since this is a $Gal(X_2/X_1)$-invariant direct factor of $f_{2*}\omega_{(X_2,A)/(S,L)}$ that is weakly positive. Since there is a generic isomorphism

$$f_{1*}\omega_{(X_1,A)/(S,L)}(\lceil\omega^{\otimes m-1}_{(X,A)/(S,L)}\otimes H^{\otimes r(m-1)}(-\frac{m-1}{m}D)\rceil) \to f_{1*}\omega_{(X_1,A)/(S,L)}(\omega^{\otimes m-1}_{(X,A)/(S,L)}\otimes H^{\otimes r(m-1)}),$$

the latter term is also weakly positive. Hence this $Gal(X_1/X)$-invariant direct factor

$$f_*\omega_{(X,A)/(S,L)}(\omega^{\otimes m-1}_{(X,A)/(S,L)}\otimes H^{\otimes r(m-1)})$$

is also weakly positive. Therefore one has

$$(r-1)m-1 < r(m-1),$$

that is $r \leq m$. Hence

$$f_*\omega_{(X,A)/(S,L)}(\omega^{\otimes m-1}_{(X,A)/(S,L)}\otimes H^{\otimes m^2-1})$$

is weakly positive. m^2-1 and H are constants independent of choice of an ample invertible sheaf H. Thus Viehweg's lemma([Vieh]) implies that

$$f_*\omega^{\otimes m}_{(X,A)/(S,L)}$$

is weakly positive.

□

3 Abundance of a canonical divisor

If $\dim Y < \dim X$ and $\omega_{(Y,A)}$ is weakly positive, there exists a variety Y_m which is birationally equivalent to Y and which has only $\mathbb{Q}$-factorial and terminal singularities such that $\omega_{(Y_m,A_{Y_m})}$ is abundant by induction argument. Minimal model program impies that

If ω_Y is weakly positive, one has $\kappa(\omega_Y)\geq 0$.

If $\omega_{(Y,A)}$ is not weakly positive, Y is uniruled.

When Y is uniruled, there exists the rational quotient R such that a general fibre of Y/R is rationally connected([Ko2]).

Proposition 3.1. *If $f:(X,A)\to(S,L)$ is a fibre space, $\kappa(\omega_{(X,A)/(S,L)})\geq 0$ and $\dim X/S=1$, then the canonical homomorphism $f^*f_*\omega^{\otimes m}_{(X,A)/(S,L)}\to\omega^{\otimes m}_{(X,A)/(S,L)}$ $(m>0)$ is surjective and $\omega_{(X,A)/(S,L)}$ is f-abundant.*

Proof. Since $\kappa(\omega_{(X,A)/(S,L)}) \geq 0$, $\dim X/S = 1$, a general fibre of $\omega_{(X,A)/(S,L)}$ is abundant. $R^1 f_* \omega^{\otimes m}_{(X,A)/(S,L)}$ is torsion free(Nakayama([Nak])). Hence rank$f^* f_* \omega^{\otimes m}_{(X,A)/(S,L)}$ is constant over each point of a curve which passes through arbitraly two points of S. $f_* \omega^{\otimes m}_{(X,A)/(S,L)}$ is compatible for base change. Since $\dim X/S = 1$, $\omega_{(X,A)/(S,L)}$ is f-abundant. □

Corollary 3.2. *Let $f : (X, A) \to (S, L)$ be a fibre space and assume that there exists a $\mathbb{Q}$-big invertible sheaf L such that $\kappa(\omega_{(X,A)/(S,L)}) \geq 0$, $\kappa(\omega_{(S,L)}) \geq 0$ and by induction assumption, $\omega_{(S,L)}$ is nef(resp. abundant), then $\omega_{(X,A)}$ is nef(resp. abundant).*

Proof. Let C be a curve on X. If $f(C)$ is a point, the intersection number of C and $\omega_{(X,A)/(S,L)}$ is non negative since $\omega_{(X,A)/(S,L)}$ is f-abundant. If $f(C)$ is a curve, then $f_* \omega^{\otimes m}_{(X,A)/(S,L)}$ $(m > 0)$ is nef. Hence the intersection number between C and $\omega_{(X,A)/(S,L)}$ is non negative. Hence that of C and $\omega_{(X,A)}$ is also non negative. □

Lemma 3.3. *Let $X/\mathbb{P}^1$ be a fibre space. Assume that $\omega_{(X,A)}$ is weakly positive. Then $\kappa(\omega_X) \geq 0$*

Proof. There exists a $\mathbb{Q}$-big sheaf L such that $\kappa(\omega_{(X,A)/(S,L)}) \geq 0$, $\kappa(\omega_{(S,L)}) \geq 0$ for a fibre space $f : X \to \mathbb{P}^1$ and there exists an integer $m > 0$ such that $f_* \omega^{\otimes m}_{(X,A)/(S,L)}$ is weakly positive. That is, $f_* \omega^{\otimes m}_{(X,A)}$ is weakly positive for some $m > 0$.

There exists the canonical homomorphism $f^* f_* \omega^{\otimes m}_{(X,A)} \to \omega^{\otimes m}_{(X,A)}$. Since $f^* f_* \omega^{\otimes m}_{(X,A)} = \oplus_k \mathcal{O}_{\mathbb{P}^1}(j_k)$, $j_k \geq 0$, $\kappa(\omega_{(X,A)}) \geq 0$. □

Theorem 3.4. *Let X be a non singular projective variety. If $\omega_{(X,A)}$ is weakly positive, then $\kappa(\omega_{(X,A)}) \geq 0$.*

Proof. Emmbed X into a projective space and choose a suitable center in the projective space. Desingularizing X, one has a Lefschetz pencil. □

Theorem 3.5. *Let $f : X \to S$ be a fibre space with $\omega_{(X,A)}$ weakly positive and* $\mathrm{var}(X/S) = \dim S$. *Then there exists a big $\mathbb{Q}$ invertible sheaf L over S such that $\kappa(\otimes\omega_{(X,A)/(S,L)}) \geq 0$ and $\kappa(\omega_{(S,L)}) \geq 0$.*

Proof. Since $\omega_{(X,A)}$ weakly positive, $\kappa(\omega_{(X,A)}) \geq 0$. Iitaka's lemma implies that $\kappa(\omega_{(X,A)|\overline{\eta}}) \geq 0$ for the geometric generic point $\overline{\eta}$ of X. By inductive assumption the geometric generic fibre has a model satisfying an abundant conjecture. By Kawamata's theorem and its version there exists a number $m >$ such that $\kappa(\det f_* \omega^{\otimes m}_{(X,A)/S}) \geq \mathrm{var}(X/S)$. Hence one can find a big $\mathbb{Q}$ invertible sheaf L over S such that $\kappa(\omega_{(X,A)/(S,L)})$ and $\kappa(\omega_{(S,L)} \geq 0$ since $\kappa(\omega_{(X,A)}) \geq 0$. Taking a relative Lefschetz pencil from X/S repeatedly, one obtains a fibre space X/S_1 of $\dim X/S_1 = 1$. Then there exists a big $\mathbb{Q}$ sheaf L_1 over S_1 such that $\kappa(\omega_{(X,A)/(S_1,L_1)})$ and $\kappa(\omega_{(S_1,L_1)} \geq 0$. Choose an abundant model S_1 by inductive hypothesis. Then one can find an abundant model X_1 birationally equivalent to X by applying lemmas above for curves family X_1/S_1. □

3.1 Uniruledness

Lemma 3.6 (Rationality)**.** *Assume that* K_X *is not weakly positive and that* D *is big. Then* $t_0 = \inf\{0 < t \in \mathbb{Q}; K_X + tD \textit{is weakly positive}\}$ *is a rational number. Furthermore,* $\dim X > \kappa(K_X + t_0 D) \geq 0$.

Proof. Put $t_0 = \inf\{0 < t \in \mathbb{Q}; K_X + tD\text{is weakly positive}\}$. If t_0 is a rational number, the lemma above implies $\kappa(K_X + t_0 D) \geq 0$. If $\kappa(K_X + t_0 D) = \dim X$, then Kodaira's lemma implies that there exists an $0 < \varepsilon \in \mathbb{Q}$ such that $\kappa(K_X + t_0 D - \varepsilon D) \geq 0$, which is a contradiction. Hence $\dim X > \kappa(K_X + t_0 D) \geq 0$.
Assume that $t_0 \notin \mathbb{Q}$. Approximate a rational number t_2 such that $t_2 > t_0$. Then $\kappa(K_X + t_2 D) = \dim X$. Put $D' = \varepsilon(K_X + t_2 D) + D$ for a rational number $\varepsilon > 0$. Set $\tau_0 = \inf\{0 < t \in \mathbb{Q}; K_X + tD'\}$. Clearly $\tau_0 < t_0$. Hence there exists a rational number t_3 such that $\tau_0 < t_3 < t_0$. Since $\kappa(K_X + t_3 D') = \kappa(K_X + t_3(\varepsilon(K_X + t_2 D) + D) = \dim X$ and since $K_X + t_3(\varepsilon(K_X + t_2 D) + D) = (1 + t_3\varepsilon)K_X + t_3(\varepsilon t_2 + 1)D$, one has the ratio between the coefficients of K_X and D, $\tau = \dfrac{t_3(1 + \varepsilon t_2)}{1 + t_3\varepsilon}$. If $0 < \varepsilon \in \mathbb{Q}$ tends to zero, this ratio tends to $\tau \to t_3$. Hence there exists a rational number $\varepsilon > 0$ such that $\tau < t_0$. This is a contradiction. □

References

[AM] Adem, A. and Milgram, R., *Cohomology of finite groups.*, Grundlehren math. Wiss. 309, p. 317 (1991).

[BBD] Beilinson A.A., Bernstein I.N., Deligne P., *Faisceaux pervers.*, Analyse et Topologie sur les espaces singuliers(I), Conférence de Luminy, juillet 1981, Astérisque 100(1982).

[Berth] Berthelot, P., *Altérations de variétés algébraiques d'après A.J. DE JONG*, Séminaire Bourbaki 48, $n°$ 815, pp.273-311(1997).

[BJ] Borceux, F. and Janelidze, G. *Galois Theories.*, cambridge studies in advanced math. 72 pp.341 ISBN 0 521 80309 8 2001

[Breen1] Breen, L., *Théorie de Schreier supériure.*, Ann. scient. Éc. Norm. Sup., 4e série, t. 25, pp. 465-514(1992).

[Breen2] Breen, L., *On the classification of 2-gerbes and 2-stacks*, 225 Astérisque Société Mathématique p. 160 (1994).

[Fuj] Fujita, T., *On Kähler fibre spaces over curves.*, J. Math.Soc. Japan 30, pp. 779-794(1978).

[Gir] Giraud, J., *Cohomologie non abélienne.*, Grundlehren math. Wiss., Springer-Verlag Berlin Heidelberg New York, p. 467,(1971).

[SGA] Grothendieck, A. et al., *Sémiaire de géométrie algébrique du Bois-Marie*, SGA1, SGA4 I,II,III, SGA41/2, SGA5, SGA7 I,II, Lecture Notes in Mat., vols. 224,269-270-305,569,589,288-340, Springer-Verlag, New York, 1971-1977.

[GG] Grothendieck, A., *Fondaments de la géométrie algébrique.*, Secrétrariat mathématique, 11 rue Pierre Curis, Paris 5e, p. 236 (1962).

[Dix] Grothendieck A., Giraud J., Kleiman S., Raynaud M., Tate J., *Dix exposés sur la cohomologie des schémas*, Advanced studies in pure math., vol. 3, North-Holland, Amsterdam, Paris, 1968, pp.386.

[HS] Hirshowits, A. and Simpson, C., *Descente pour les n-champs.*, arXiv:math. AG/9807049v3 13 Mar 2001

[Iita] Iitaka, S., *Introduction to birational geometry.*, Graduate Textbook in Mathematics, Springer-Verlag, p. 357 (1976).

[Laum] Laumon, G., Moret-Bailly, L., *Champs algébraiques.*, Université de Paris-sud Mathématiques, p. 94 (1992).

[Kato] Kato, K., *Logarithmic structures of Fontaine-Illusie.*, Algebraic Analysis, Geometry and Number Theory., The Johns-Hopkins Univ. Press, Baltimore, MD, pp. 191-224(1989).

[Kaw] Kawamata, Y., *Minimal models and the Kodaira dimension of algebraic fibre spaces.*, J. Reine Angew. Math. 363, pp. 1-46 (1985).

[1] KKMS Kemf, G., Knudsen, F., Mumford, D., Saint-Donat, B., *Troidal embeddings I.*, Springer Lecture Notes, 339(1973).

[Ko1] Kollár, J., *Shafarevich maps and automorphic forms.*, Princeton university press, Princeton, New Jersey pp.201 1995.

[Ko2] Kollár, J., *Rational curves on algebraic varieties*, Springer, Berlin-Heiderberg-Newyork-Tokyo, (1995)

[Km1] Maehara, K., *Diophantine problems of algebraic varieties and Hodge theory in International Symposium Holomorphic Mappings, Diophantine Geometry and related Topics in Honor of Professor Shoshichi Kobayashi on his 60th birthday.* R.I.M.S., Kyoto University October 26-30, Organizer: Junjiro Noguchi(T.I.T.), pp. 167-187 (1992).

[Km2] Maehara, K., *Algebraic champs and Kummer coverings.* A.C.Rep. T.I.P. Vol.18, No.1, pp.1-9(1995).

[Km3] Maehara, K., *Conjectures on birational geometry.* A.C.Rep. T.I.P. Vol.24, No.1, pp.1-10(2001).

[Mats] Matsuki K., Introduction to the Mori Program, Universitext p. 468 Springer 2000

[MP] Miyaoka Y., Peternel T., Geometry of Higher Dimensional Algebraic Varieties DMV Seminar Band 26 Birkhäiser p. 213 1997

[Mch] Mochizuki S., The local Pro-p Anabelian Geometry of Curves, Research Institute for Mathematical Sciences, Kyoto University RIMS-1097(1996).

[Mum] Mumford D., Selected Papers: On the classification of varieties and moduli spaces Springer p. 795 2003

[Nak] Nakayama N., Invariance of the plurigenera of algebraic varieties under minimal model conjectures, Topology 25(1986) 237-251.

[Ray] Raynaud, M., Gruson, L., Critères de platitude et de projectivité., Inventiones Math,. 13, pp.1-89(1971).

[RBZL] Ribes L., Zalesskii P., Profinite Groups Ergebnisse der Mathematik und ihrer Grenzgebiete 3.Folge Vol.40 Springer(1991).

[Ws] Schmid, W., *Variation of Hodge structure: the singularities of the period mapping.*, Inventiones math., 22., pp. 211-319(1973).

[2] J-P. Serre. Cohomologie Galoisienne. Lecture Notes in Mathematics. , **5** 4th Ed. 1973.

[S1] Simpson, C., *Algebraic (geometric) n-stacks.*, arXiv:math. AG/9609014

[S2] Simpson, C., *A closed model structure for n-categories, internal Hom, n-stacks and generalized Seifer-Van Kampen.* arXiv:math: AG/9704006 v2 17 Mar 2000

[Shatz] Shatz, S., *Profinite groups, arithmetic, and geometry.* Annals of Math. studies, N. 67, Princeton Uni. Press, p. 252,(1972).

[Vieh] Viehweg, E. *Quasi-projective Moduli for Polarized Manifolds.* Ergebnisse der Mathematik und ihrer Grenzgebiete, 3.Folge.Band 30, p. 320 (1991).

[Zuo] Zuo, K., Representations of fundamental groups of algebraic varieties., Lecture Notes in Math. 1708, Springer, ISBN 3-540-66312-6.