Capture components for cubic polynomials with parabolic fixed points

Shizuo Nakane*

The parameter space for a family of cubic polynomials with parabolic fixed points of multiplier one is investigated. Especially, the dynamics on the boundaries of the capture components is revealed.

1 Introduction

In this note, we will investigate the dynamics of the family

$$Per_1(1) : P_a(z) = z^3 + az^2 + z, \quad a \in \mathbb{C}.$$

of cubic polynomials with parabolic fixed point 0 of multiplier one. Especially we will study the paramater space of our family. One of the two critical points of the map P_a belongs to the immediate basin $\mathcal{B}_a^*(0)$ of the parabolic fixed point 0. The dynamics of P_a is completely determined by the behaviour of the orbit of another critical point. We are much interested in the parameters for which both critical points belong to the basin $\mathcal{B}_a(0)$. We call here the set of such parameters the *parabolic set* and its connected component a *parabolic component*.

Our family $Per_1(1)$ in cubic polynomials has been investigated by many authors. Douady-Hubbard [DH] studied the discontinuity of the straightening map of cubic-like maps on $Per_1(1)$. Milnor [M1] considered the family of real cubic polynomials and conjectured the non-local connectivity of the cubic connectedness locus. Lavaurs [L] settled this conjecture through the study of $Per_1(1)$. See also Epstein-Yampolsky [EY]. Thus $Per_1(1)$ reflects the features of the cubic dynamics much different from the quadratic one.

Willumsen [W] gave necessary conditions for stretching rays to accumulate on a map in the main parabolic component of $Per_1(1)$. Inspired by her work, the author showed, in the joint work [KN] with Y. Komori that, in a certain region of the space of real cubic polynomials, most stretching rays have non-trivial accumulation sets on the real slice of the main parabolic component of $Per_1(1)$. In fact, they oscillate wildly as they approach

*Professor, General Education and Research Center, Tokyo Polytechnic University,
Received Sept. 8, 2005

$Per_1(1)$. In order to study stretching rays in non-real regions, we have to reveal the structure of those parabolic components. This note is a first step toward this aim. And we will investigate the dynamics on the boundaries of the capture components.

2 Connectedness locus

Note that critical points of the map P_a :

$$c_\pm(a) = \frac{-a \pm \sqrt{a^2-3}}{3} = \frac{a}{3}\left(-1 \pm \sqrt{1-3/a^2}\right).$$

are two branches of a two-valued holomorphic function on $\overline{\mathbb{C}}$. It branches at $a = \pm\sqrt{3}$. Or they are holomorphic functions on the double covering space of $\overline{\mathbb{C}} - \{\pm\sqrt{3}\}$. In order to consider them on $\overline{\mathbb{C}}$, we must fix their branches. Since $c_\pm(a)$ are holomorphic for large $|a|$, we fix such branches. Then they are single-valued holomorphic in $\overline{\mathbb{C}} - [-\sqrt{3}, \sqrt{3}]$. Note that they are replaced by each other when we go through the slit $[-\sqrt{3}, \sqrt{3}]$. On this slit, both critical points belong to $\mathcal{B}^*_a(0)$. Thus the slit is included in the connectedness locus. Since they have the following asymptotic behaviours near $a = \infty$:

$$c_+(a) = -\frac{1}{2a} + O(\frac{1}{a^3}), \quad c_-(a) = -\frac{2a}{3} + O(\frac{1}{a}),$$

the critical values satisfy

$$P_a(c_+(a)) = O(\frac{1}{a}), \quad P_a(c_-(a)) \approx \frac{4a^3}{27}.$$

Thus, for large $|a|$, $c_-(a)$ escapes to ∞ and $c_+(a)$ is contained in $\mathcal{B}^*_a(0)$.

Lemma 2.1. *The connectedness locus $M_1(1)$ of the family $Per_1(1)$ is characterized by*

$$M_1(1) = \{a \in \mathbb{C}; c_-(a) \in K_a = K(P_a)\}.$$

In $\mathbb{C} - M_1(1)$, the unique indifferent cycle 0 is persistent. Hence P_a is J-stable there. Actually we have

Lemma 2.2. *The complement of $\partial M_1(1)$ is the set of parameter a such that P_a is J-stable.*

proof. Note that P_a is J-stable if and only if the family $\{P_a^k(c_\pm(a); k \geq 0\}$ forms a normal family in a neighborhood of a. If we put $m_a = 3max(|a|, 1)$, it follows $K_a \subset \mathbb{D}_{m_a}$. Thus, if $a \in IntM_1(1)$, both critical points are contained in $\mathbb{D}_{m_a}$ and P_a is J-stable by

Montel's theorem. If $a \in \partial M_1(1)$, P_a is not J-stable since $\{P_a^k(c_-(a))\}$ is not normal. This completes the proof. □

Now let $\tilde{c}_-(a)$ be the co-critical point of $c_-(a)$. It satisfies $P_a(\tilde{c}_-(a)) = P_a(c_-(a))$. If φ_a denotes the Böttcher coordinate of P_a, $\varphi_a(\tilde{c}_-(a))^3 = \varphi_a(P_a(\tilde{c}_-(a))) \approx 4a^3/27$. Thus, if we put $\Phi(a) = 3\varphi_a(\tilde{c}_-(a))/\sqrt[3]{4}$, we have

Proposition 2.1. $\Phi : \overline{\mathbb{C}} - M_1(1) \to \overline{\mathbb{C}} - \overline{\mathbb{D}_{3/\sqrt[3]{4}}}$ *is a conformal isomorphism satisfying* $\lim_{a\to\infty} \Phi(a)/a = 1$.

Corollary 2.1. $M_1(1)$ *is connected.*

Now, external rays are defined for $M_1(1)$ and we can discuss their landing properties. Note that the correspondence between the parameter space and the dynamical plane is done through the co-critical point.

3 Conformal position maps

In this section, we will show that every parabolic components are simply connected. For a parabolic component W, the critical point $c_+(a)$ always belongs to $\mathcal{B}_a^*(0)$ and there exists $k \geq 0$ such that $P_a^k(c_-(a))$ first hits $\mathcal{B}_a^*(0)$. We call such k the *preperiod* of W. W is called a *capture* component if $k > 0$. If $k = 0$, that is, if both critical points are contained in $\mathcal{B}_a^*(0)$, W is called an *adjacent* component. It turns out that there are only two adjacent components, each containing $\sqrt{3}$ or $-\sqrt{3}$.

Let W be a parabolic component of preperiod k. In order to study the global topology of parabolic components, we use the *conformal position map* m after Zakeri [Z]. Let $\psi_a : \mathbb{D} \to \mathcal{B}_a^*(0)$ be the Riemann map satisfying $\psi_a(0) = c_+(a)$, $\psi_a(1) = 0$. We define $m : W \to \mathbb{D}$ by $m(a) = \psi_a^{-1}(P_a^k(c_-(a)))$.

Lemma 3.1. *For any capture component* W*,* ψ_a *depends holomorphically on* $a \in W$*.*

proof. Consider the map $R_a = \psi_a^{-1} \circ P_a \circ \psi_a : \mathbb{D} \to \mathbb{D}$. Since it is a proper holomorphic map between $\mathbb{D}$, it is a Blaschke product of degree two with critical points 0 (and ∞). Since J_a is locally connected, ψ_a can be continued to a continuous map $\overline{\mathbb{D}} \to \overline{\mathcal{B}_a^*(0)}$. Because of the maximum principle, it must be injective on $\partial\mathbb{D}$. Thus $\partial\mathcal{B}_a^*(0)$ is a Jordan curve and $\psi_a : \overline{\mathbb{D}} \to \overline{\mathcal{B}_a^*(0)}$ is an onto homeomorphism. Then ψ_a conjugates P_a to R_a also on $\partial\mathbb{D}$. Hence $J(R_a) = \partial\mathbb{D}$ and 1 is a fixed point. Since points on $\partial\mathbb{D}$ are not attracted by 1, 1 is not attracting but parabolic with multiplier one. Since Blaschke product is commutable with the map $z \mapsto 1/\overline{z}$, $\overline{\mathbb{C}} - \overline{\mathbb{D}}$ is also the basin of 1 and 1 has multiplicity three. That is, $R_a''(1) = 0$. Now it is easy to see that $R_a(z) \equiv R(z) = \dfrac{z^2 + 1/3}{1 + z^2/3}$, which is independent of

a. ψ_a maps each point on the inverse orbit by R of 1 to the point on the inverse orbit by P_a of 0, which moves holomorphically on a. Thus $\psi_a \circ \psi_{a_0}^{-1}$ gives a holomorphic motion of the inverse orbit of 0. By the λ-lemma, it continues to to a holomorphic motion of $\partial\mathcal{B}_a^*(0)$. Hence, $\psi_a \circ \psi_{a_0}^{-1}$ on $\partial\mathcal{B}_a^*(0)$, and consequently ψ_a on $\partial\mathbb{D}$ depends holomorphically on a. Now, by the Poisson formula, ψ_a on $\mathbb{D}$ depends holomorphicall on a. □

Thus the map $m : W \to \mathbb{D}$ is holomorphic for any capture component.

Lemma 3.2. *For any capture component W of preperiod k, $m : W \to \mathbb{D}$ is proper.*

proof. Suppose $a_n \in W$ tends to $a_0 \in \partial W$ and $m(a_n) \to m_0 \in \mathbb{D}$. By Montel's theorem, we may assume ψ_{a_n} converges to a conformal map ψ_0 locally uniformly on $\mathbb{D}$.

Then $\psi_0(\overline{\mathbb{D}_r}) \subset \mathcal{B}_{a_0}^*(0)$ for any $r < 1$. In fact, if there exist an $r < 1$ and a point $w_1 \in \psi_0(\overline{\mathbb{D}_r}) \cap J_{a_0}$, then, for any $r' > r$, $\psi_0(\mathbb{D}_{r'})$ is an open neighborhood of w_1 and contains a repelling periodic point w_0 of P_{a_0}. This point is locally holomorphically continued to a repelling periodic point $w(a)$ of P_a. Thus $w(a) \in J_a \cap \psi_0(\mathbb{D}_{r'})$ for any a close to a_0. On the other hand, since $\psi_{a_n}(\overline{\mathbb{D}_{r''}}) \to \psi_0(\overline{\mathbb{D}_{r''}})$ for any $r'' > r'$, $\psi_0(\overline{\mathbb{D}_{r'}}) \subset \psi_{a_n}(\overline{\mathbb{D}_{r''}})$ holds for large n. Thus J_{a_n} does not intersect $\psi_0(\overline{\mathbb{D}_{r'}})$ for large n. This is a contradiction. Thus $\psi_0(\mathbb{D}) \subset \mathcal{B}_a^*(0)$.

Now $P_{a_0}^k(c_-(a_0)) = \lim_{n\to\infty} \psi_{a_n}(m(a_n)) = \psi_0(m_0) \in \mathcal{B}_{a_0}^*(0)$, which implies a_0 is parabolic. This contradicts the fact $a_0 \in \partial W$. This completes the proof. □

Lemma 3.3. *Let W be a capture component of preperiod k and $a, b \in W$. Suppose a qc-map $\varphi : \mathbb{C} \to \mathbb{C}$ conjugates P_a on $\mathbb{C} - \mathcal{B}_a(0)$ to P_b on $\mathbb{C} - \mathcal{B}_b(0)$. If $m(a) = m(b)$, then there exists a qc-conjugacy ψ on $\mathbb{C}$ between P_a and P_b such that ψ is conformal on $\mathcal{B}_a(0)$ and coincides with φ on $\mathbb{C} - \mathcal{B}_a(0)$.*

proof. Since $R_a \equiv R$, the map $\psi = \psi_b \circ \psi_a^{-1}$ gives a conformal equivalence betwee P_a on $\mathcal{B}_a^*(0)$ and P_b on $\mathcal{B}_b^*(0)$. From the assumption $m(a) = m(b)$, we have $\psi(P_a^k(c_-(a))) = P_b^k(c_-(b))$. Hence ψ can be holomorphically continued to $\mathcal{B}_a(0)$ by $\psi = P_b^{-n} \circ \psi \circ P_a^n$. Recall that the maps ψ_a and ψ_b can be extended to the homeomorphisms from $\overline{\mathbb{D}}$ onto $\overline{\mathcal{B}_a^*(0)}$ and $\overline{\mathcal{B}_b^*(0)}$ respectively. Since $\psi = \varphi$ on the inverse orbit of 0, $\psi = \varphi$ holds also on $\partial\mathcal{B}_a^*(0)$. Pulling back by P_a, the same holds on $J_a = \partial\mathcal{B}_a(0)$. If we extend ψ to the complemen of K_a by $\psi = \varphi$, $\psi : \mathbb{C} \to \mathbb{C}$ is a homeomorphism. Then ψ is a qc-map by Rickman's theorem. □

Proposition 3.1. *For every capture component of preperiod k, $m : W \to \mathbb{D}$ is a conformal isomorphism.*

proof. We have only to show the injectivity of m. Suppose $m(a) = m(b)$. Using Böttcher coordinates, the conformal equivalence $\varphi_{a,c} = \varphi_c^{-1} \circ \varphi_a$ gives a holomorphic motion of $\mathbb{C} - K_a$. By the optimal λ-lemma of Slodkowski [Sl], this can be extended to the holomorphic motion $\overline{\varphi_{a,c}}$ of $\mathbb{C}$. We apply the previous lemma to $\varphi = \overline{\varphi_{a,b}}$. Then there exists a qc-conjugacy ψ between P_a and P_b on $\mathbb{C}$, conformal on $\mathcal{B}_a(0)$ and coincides with φ on $\mathbb{C} - \mathcal{B}_a(0)$. Since ψ is a qc-map on $\mathbb{C}$ and conformal except on the measure zero set J_a, it is conformal on $\mathbb{C}$. Thus P_a is conformally conjugate to P_b on $\mathbb{C}$. Then $a = b$. This completes the proof. □

Corollary 3.1. *Every capture component is simply connected.*

4 Boundaries of capture components

In this section, we give some properties of the boundaries of capture components.

Lemma 4.1. *Suppose $a_0 \neq 0$. If the external ray $R_{a_0}(t)$ of P_{a_0} with angle $t = p/3^k$ or $t = p/(2 \cdot 3^k)$ lands at $z_0 \in J_{a_0}$ and $P_{a_0}^n(z_0)$ is not a critical point for any $n \geq 0$, then there exists an open neighborhood U of a_0 such that, for any $a \in U$, $R_a(t)$ lands at a repelling or parabolic preperiodic point z_a. The landing point z_a depends holomorphically on a in U.*

Lemma 4.2. *The external rays $R_M(t)$ of $M_1(1)$ with angles $t = \pm 1/6, \pm 1/3$ land at the origin.*

proof. The external rays $R_a(0)$ and $R_a(1/2)$ land at fixed points of P_a if they do not meet the critical point $c_-(a)$. If $R_a(0)$ (resp. $R_a(1/2)$) meets $c_-(a)$, then one of $R_a(\pm 1/3)$ (resp. one of $R_a(\pm 1/6)$) meets the co-critical point $\tilde{c}_-(a)$, i.e. a lies on one of $R_M(\pm 1/3)$ (resp. one of $R_M(\pm 1/6)$). In other words, $R_a(0)$ (resp. $R_a(1/2)$) lands at a fixed point 0 or $-a$ unless a belongs to $R_M(\pm 1/3)$ (resp. $R_M(\pm 1/6)$). Thus, at the accumulation point a_0 of $R_M(\pm 1/3)$ (resp. $R_M(\pm 1/6)$), the landing of $R_a(0)$ (resp. $R_a(1/2)$) is unstable. On the other hand, Lemma 4.1 implies those stabilities at $a_0 \neq 0$. Thus those rays must land at the origin. □

The four rays $R_M(\pm 1/3)$ and $R_M(\pm 1/6)$ and their landing point 0 separate the parameter space into four parts. In the region $\mathcal{R}_1$ bounded by $R_M(-1/6)$ and $R_M(1/6)$, $R_M(0)$ (resp. $R_M(1/2)$) lands at 0 (resp. $-a$). In the region $\mathcal{R}_3$ bounded by $R_M(1/3)$ and $R_M(-1/3)$, $R_M(1/2)$ (resp. $R_M(0)$) lands at 0 (resp. $-a$). In the remaining two regions $\mathcal{R}_2$ and $\mathcal{R}_4$, $R_M(0)$ and $R_M(1/2)$ land at 0.

Lemma 4.3. *For a point a on the boundary of a parabolic component W, P_a has neither Siegel disks nor Cremer cycles.*

proof. If P_{a_0} has a Siegel or Cremer periodic point z_0, then evidently $a_0 \neq 0$. By Lemma 3.3 of Kiwi [K], z_0 and $\mathcal{B}^*_{a_0}(0)$ are separated by a union $\mathcal{R}$ of a finite collection of closed preperiodic external rays and $\mathcal{R}$ separates the orbits of critical points. Note that preperiodic external rays must land at repelling or parabolic preperiodic points. Since P_{a_0} has no other non-repelling cycles, the landing points of the rays in $\mathcal{R}$ are repelling except at 0. By the previous lemma, landing of those rays is stable around a_0. Hence, the same holds for $a \in W$ close to a_0. But, in W, the orbit of $c_-(a)$ hits $\mathcal{B}_a^*(0)$ after finite iteration. This is a contradiction. □

We will show that, if W is capture, P_a has no parabolic cycles except 0. Suppose W is capture and $a \in W$. Since K_a is pathwise connected, there is a path γ connecting $c_-(a)$ to $c_+(a)$. We denote z_a the point where γ first hits $\partial\mathcal{B}_a^*(0)$. Since K_a is full, z_a is uniquely determined independent of the choice of γ. Since K_a is locally connected, at least two external rays land at z_a. Among them, we take two rays $R_a(t_1)$ and $R_a(t_2)$ separating $c_-(a)$ from $\mathcal{B}_a^*(0)$ and consider the sector S_0 bounded by these two rays and z_a, containing $c_-(a)$. In the following, we use the theory of orbit portraits developed in Milnor [M2]. We denote by $A_a(z)$ the set of angles of external rays of P_a landing at z.

Lemma 4.4. *Let W be a capture component of preperiod k and $a \in W$. Then z_a is a periodic point of period $m \leq k$.*

proof. Suppose z_a is not periodic. Put $z_j = P_a^j(z_a)$ and let S_j be the succesive image sectors of S_0 at z_j bounded by $R_a(3^j t_1)$ and $R_a(3^j t_2)$. (Note that $P_a(S_0)$ covers $\mathbb{C}$ and doubly covers S_1.) Then, since S_j contains no critical points, it does not intersect $\mathcal{B}_a^*(0)$ for any $j \geq 1$. But its angular length $3^j(t_2 - t_1)$ eventually becomes greater than one, a contradiction. □

Apparently z_a is repelling unless it is 0. If $W \subset \mathcal{R}_1$ or $W \subset \mathcal{R}_3$, $z_a \neq 0$ for any $a \in \overline{W}$.

Lemma 4.5. *The point z_a is repelling also on ∂W unless it is* 0.

proof. Suppose z_{a_0} is not repelling for some $a_0 \in \partial W$. It must be parabolic by Lemma 4.3. Then, as $a \in W$ tends to a_0, z_a meets other repelling periodic points, say $z_{a,j}$, $1 \leq j \leq k$. Then $A_{a_0}(z_{a_0})$ is the union of $A_a(z_a)$ and $A_a(z_{a,j})$. By the theory of orbit portraits, this happens only if $k = 1$ and the combinatorial rotation number at z_{a_0} is 0, i.e. z_{a_0} has just two angles. This contradicts the fact that z_a has at least two angles. □

The above proof implies that, if $z_a \neq 0$ for $a \in W$, the same holds also for $a \in \partial W$.

Lemma 4.6. *Suppose $z_a \neq 0$ for $a \in W$. Then $A_{a_0}(z_{a_0}) = A_a(z_a)$ holds for $a_0 \in \partial W$ and $a \in W$.*

proof. It follows $z_{a_0} \neq 0$. Since z_{a_0} is repelling by Lemma 4.5, so is z_a for any $a \in W$ close to a_0. By stability, it follows $A_{a_0}(z_{a_0}) \subset A_a(z_a)$. If $A_{a_0}(z_{a_0}) \neq A_a(z_a)$, there exists $t \in A_a(z_a)$ such that $R_{a_0}(t)$ lands at some point $w_{a_0} \neq z_{a_0}$. By stability, w_{a_0} must be parabolic. Evidently $w_{a_0} \neq 0$. Then, for $a \in W$, the corresponding point $w_a \neq z_a$ is repelling and has angle t, a contradiction. □

From the three lemmas above, it follows that, if $z_a \neq 0$ in W, the rays landing at z_a separate $c_-(a)$ and $\mathcal{B}_a^*(0)$ for any a in a neighborhood W' of $\overline{W}$. Since $c_-(a) \neq z_a$, we conclude that $c_-(a)$ does not belong to an open neighborhood U_a of $\overline{\mathcal{B}_a^*(0)}$ for any $a \in W'$. Put $U_a' = U_a - \cup_{j=0}^{m-1}\overline{S_j}$ and $U_a'' = U_a \cap P_a^{-1}(U_a) - \cup_{j=0}^{m-1} P_a^{-1}(\overline{S_j})$. Then $P_a : U_a'' \to U_a'$ is proper holomorphic. By thickening, we get a quadratic-like map $P_a : V_a \to V_a'$. By straightening, this map is hybrid equivalent to a quadratic polynomial p. Since P_a has a parabolic fixed point 0 of multiplier one, $p(z) = z^2 + 1/4$.

Capture component W where $z_a = 0$ sits in the region $\mathcal{R}_2$ or $\mathcal{R}_4$. In this case, two rays $R_a(0)$ and $R_a(1/2)$ stably lands at $z_a = 0$. Another fixed point $z_0 = -a$ is separated by these two rays from $c_+(a)$. We take a path γ in K_a connecting z_0 to 0.

Lemma 4.7. *There exists a sequence of points on the inverse orbit of z_0 converging to 0.*

proof. First note that γ is not included in a Fatou component. Otherwise, that component is invariant since 0 has combinatorial rotation number 0.

Suppose γ does not intersect any Fatou components. Then, since it does not contain $c_-(a)$, P_a is injective on γ. Then $P_a(\gamma) = \gamma$ since P_a fixes its endpoints. Because P_a is repelling near both endpoints, P_a must have another fixed point in the interior of γ, a contradiction.

Thus γ intersects both the Fatou set and the Julia set. The rotation number of z_0 is not 0 since $0, 1/2 \notin A_a(z_0)$. Hence, the local image of γ around z_0 is another branch. Thus there exists a preimage $z_1 \in \gamma$ of z_0. Let γ_1 be the subpath of γ connecting 0 to z_1. Suppose $P_a(\gamma_1)$ does not contain z_1. Since the regions bounded by γ and $P_a(\gamma_1)$ is included in K_a, z_1 is on the boundary of a Fatou component. Then z_0 is also on the boundary of a Fatou component U. Since U is periodic, U is a periodic Fatou component, a contradiction. Thus there exists a preimage $z_2 \in \gamma_1$ of z_1. Repeating this argument, we get a sequence $z_j \in \gamma$ on the inverse orbit of z_0. Since its accumulation point is a fixed point, it must be 0. This completes the proof. □

By the same proof of Lemma 4.5, it follows that z_0 is repelling for $a \in \partial W$. Moreover, at least two rays land at z_0. Hence the same is true for z_j. Using these rays, we get a quadratic-like map $P_a : V_a \to V_a'$, hybrid conjugate to p. We do not need thickening in this case. Especially, since $\partial\mathcal{B}_a^*(0)$ is homeomorphic to $J(p)$, we get the following.

Proposition 4.1. *Let W be a capture component. Then $\partial\mathcal{B}_a^*(0)$ is locally connected in a neighborhood W' of $\overline{W}$.*

Corollary 4.1. *Let W be a capture component. Then the Riemann map $\psi_a : \mathbb{D} \to \mathcal{B}_a^*(0)$ depends holomorphically on a in W'.*

proof. The same proof of Lemma 3.1 works since we only use the local connectivity of $\partial\mathcal{B}_a^*(0)$.

Lemma 4.8. *Let W be a capture component of preperiod k. Then $P_a^k(c_-(a)) \in \partial\mathcal{B}_a^*(0)$ if $a \in \partial W$.*

proof. Since $a \mapsto \partial\mathcal{B}_a^*(0)$ is a holomorphic motion on U, it is continuous with respect to the Hausdorff distance. For $a \in W$, we have $P_a^k(c_-(a)) \in \mathcal{B}_a^*(0)$. By continuity, $P_a^k(c_-(a)) \in \overline{\mathcal{B}_a^*(0)}$ for $a \in \partial W$. Since $P_a^k(c_-(a)) \notin \mathcal{B}_a^*(0)$ for $a \in \partial W$, the lemma follows. □

Corollary 4.2. *Let W be a capture component. Then, for $a \in \partial W$, P_a has no parabolic cycle except* 0.

Corollary 4.3. *The map $m : W \to \mathbb{D}$ extends to a continuous surjective map $m : \overline{W} \to \overline{\mathbb{D}}$. If, in addition, ∂W is locally connected, then $m : \overline{W} \to \overline{\mathbb{D}}$ is a homeomorphism.*

proof. In order to prove the surjectivity of m, we define the *internal ray* in W as the inverse image of a ray in $\mathbb{D}$ by the map m. For any point $w_0 = e^{2\pi it} \in \partial\mathbb{D}$, consider the internal ray $R_W(t) \equiv m^{-1}(\{re^{2\pi it}; 0 \le r < 1\})$. We do not know whether this ray lands or not on ∂W. Take any accumulation point $z_0 = \lim_{n\to\infty} m^{-1}(r_n e^{2\pi it}) \in \partial W$. By the continuity of m, it follows $m(z_0) = e^{2\pi it} = w_0$.

If ∂W is locally connected, m^{-1} has a continuous extension to $\overline{\mathbb{D}}$. Then m^{-1} is the inverse of m also on $\partial\mathbb{D}$. This completes the proof. □

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