

External rays for a regular polynomial endomorphism of \mathbb{C}^2 associated with Chebyshev mappings

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In this note, the dynamics of a regular polynomial map of \mathbb{C}^2 is investigated. Especially, landing points of the external rays are completely characterized.

1 Introduction

In this note, we consider the external rays of the map $F : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ of the form :

$$F(x, y) = (x^2 - 2y, y^2 - 2x).$$

External rays were first defined for polynomial maps on \mathbb{C} to investigate the combinatorial properties of the dynamics on the Julia sets. Let P be a monic centered polynomial with degree d of one variable. Let $\varphi = \varphi_P$ be its *Böttcher coordinate*, that is, a conformal map φ in a neighborhood of the point at ∞ satisfying

$$\varphi(P(z)) = \varphi(z)^d, \quad \lim_{z \rightarrow \infty} \frac{\varphi(z)}{z} = 1.$$

By this functional equation, it can be continued analytically until it meets a critical point. Especially, if $K(P)$ is connected, it extends to a conformal map $\varphi : \mathbb{C} - K(P) \rightarrow \mathbb{C} - \overline{\mathbb{D}}$. The *external ray* $R_P(\theta)$ of *external angle* θ is defined by the preimage of the ray $\{re^{2\pi i\theta}; r > 1\}$ by φ . We say it lands at a point $z \in J(P)$ if it is continued to $r > 1$ and converges to z as $r \rightarrow 1$. Recently, Bedford and Jonsson [BJ] defined external rays for regular polynomial endomorphisms of \mathbb{C}^k and established a landing property with some additional assumptions. Although the map F does not satisfy the assumptions in [BJ], we can investigate the landing property from the explicit expression of its Böttcher coordinate.

The map F has dynamically distinguished properties. For example, it is *critically finite*, that is, the union of the forward orbit of the critical set forms an analytic subset of

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\mathbb{C}^2 . This is because it is closely related to the *Chebyshev maps* of two variables. A typical example of Chebyshev maps of one variable is the quadratic polynomial $p(z) = z^2 - 2$, which is critically finite, too. A natural extension of this Chebyshev map to the two variables case is $f(z) = z^2 - 2\bar{z}$. By virtue of the distinguished properties of Chebyshev maps, Uchimura [U1] has obtained many interesting results.

Here we show why Chebyshev maps are easily analyzed. Put $z = \psi(t) = t + 1/t$. Then

$$p(\psi(t)) = (t + 1/t)^2 - 2 = t^2 + 1/t^2 = \psi(t^2).$$

Hence a branch of its inverse $\varphi = \psi^{-1}$ satisfies $\varphi(p(z)) = \varphi(z)^2$ and it gives the Böttcher coordinate of p . Then the external ray $R_p(\theta)$ is explicitly written by

$$z = \psi(re^{2\pi i\theta}) = re^{2\pi i\theta} + \frac{1}{r}e^{-2\pi i\theta}, \quad r > 1,$$

and it lands at the point

$$z_0 = e^{2\pi i\theta} + e^{-2\pi i\theta} = 2\cos 2\pi\theta.$$

Consequently, $J(p) = \{z = 2\cos 2\pi\theta; \theta \in \mathbb{T}\}$. In the sequel, we will apply this idea to the maps f and F .

2 External rays for the map f

Consider the map f studied in [U1] of the form :

$$f(z) = z^2 - 2\bar{z}.$$

The map $f : \mathbb{C} \rightarrow \mathbb{C}$ is not holomorphic but is associated with the Chebyshev maps of two variables and its dynamics is completely determined. See [U1]. Since the jacobian of f is :

$$Jac(f) = |\partial f / \partial z|^2 - |\partial f / \partial \bar{z}|^2 = 4(|z|^2 - 1),$$

its *critical set* $\mathcal{C}(f)$ is the unit circle $|z| = 1$. The *filled-in Julia set* $K(f)$, i.e., the set of points with bounded orbits, is parametrized (with some identification) as follows.

$$K(f) = \{z = z(\phi, \theta) = e^{2\pi i\phi} + e^{2\pi i\theta} + e^{-2\pi i(\phi+\theta)}; (\phi, \theta) \in \mathbb{T}^2\}. \quad (2.1)$$

And its boundary is the hypocycloid (see Figure 1) :

$$z = 2e^{2\pi i\theta} + e^{-4\pi i\theta}, \quad \theta \in \mathbb{T}.$$

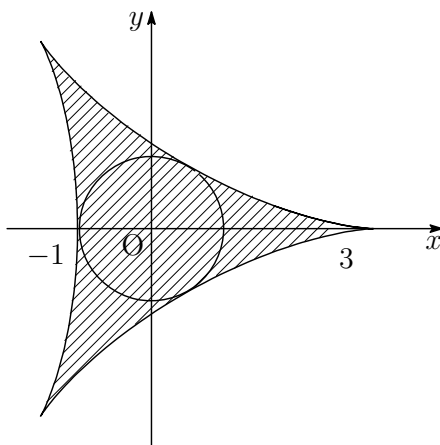


Figure 1: The dark region : $K(f)$, the circle : $\mathcal{C}(f)$

Moreover, the dynamics of f on $K(f)$ is expressed by $f(z(\phi, \theta)) = z(2\phi, 2\theta)$, which enables us to describe their dynamics by symbolic dynamics. Although this parametrization of $K(f)$ seems a bit tricky, we will give a dynamical meaning of the parameters ϕ, θ above in the next section.

First we study the Böttcher coordinate of f . Put

$$z = \psi(t) = t + \frac{1}{t} + \frac{\bar{t}}{t}.$$

Then its jacobian $Jac(\psi)$ satisfies

$$\begin{aligned} Jac(\psi) &= |z_t|^2 - |z_{\bar{t}}|^2 \\ &= \left|1 - \frac{\bar{t}}{t^2}\right|^2 - \left|\frac{1}{t} - \frac{1}{\bar{t}^2}\right|^2 \\ &= \left(1 - \frac{1}{|t|^2}\right) \left|1 - \frac{\bar{t}}{t^2}\right|^2. \end{aligned}$$

Thus ψ gives a diffeomorphism from $\mathbb{C} - \overline{\mathbb{D}}$ onto $\mathbb{C} - K(f)$ and it is easy to see

$$f(\psi(t)) = \psi(t^2), \quad \lim_{t \rightarrow \infty} \frac{\psi(t)}{t} = 1.$$

That is, the map $\varphi = \psi^{-1}$ should be the Böttcher coordinate of f and we can define the external ray as follows :

$$R_f(\theta) = \psi(\{re^{2\pi i\theta}; r > 1\}).$$

Then we have the following.

Theorem 2.1. *The external ray $R_f(\theta)$ is parametrized by*

$$z = re^{2\pi i\theta} + \frac{1}{r}e^{2\pi i\theta} + e^{-4\pi i\theta} \quad (r > 1)$$

and it lands at the point

$$z_0 = 2e^{2\pi i\theta} + e^{-4\pi i\theta} \in \partial K(f).$$

3 External rays for the map F

Now consider the following map in \mathbb{C}^2 .

$$F(x, y) = (x^2 - 2y, y^2 - 2x),$$

which is closely related to the map f in the previous section. In fact, the map F restricted to $H = \{(x, y) \in \mathbb{C}^2; y = \bar{x}\}$ is equivalent to f . Let $\mathcal{C}(F)$ be the set of the critical points of F . By a direct calculation, it follows $\mathcal{C}(F) = \{xy = 1\}$.

Let $f(z)$ be a polynomial endomorphism of \mathbb{C}^k of degree d and let $f_h(z)$ be the degree d part of $f(z)$. It is *regular* if $f_h^{-1}(0) = \{0\}$. Note that regular polynomial maps extend to analytic maps of \mathbb{P}^k . Let Π denote the hyperplane at ∞ , which is isomorphic to \mathbb{P}^{k-1} . In case $k = 2$, Π is isomorphic to the Riemann sphere $\overline{\mathbb{C}}$. For a regular polynomial map f , we denote the filled-in Julia set also by $K(f)$. It is a compact subset of \mathbb{C}^k . And $J(f)$ denotes the smallest Julia set of f , that is, the support of $\mu = (dd^c G_f)^k$. Here G_f is the Green function of f . And we put $f_\Pi = f|_\Pi$, $J_\Pi = J(f_\Pi)$.

Let $W^s(J_\Pi, f)$ be the stable set of J_Π :

$$W^s(J_\Pi, f) = \{z \in \mathbb{P}^k; \lim_{n \rightarrow \infty} \text{dist}(f^n(z), J_\Pi) = 0\}.$$

The inverse Böttcher coordinate Ψ is a homeomorphism $W_{loc}^s(J_\Pi, f_h) \rightarrow W_{loc}^s(J_\Pi, f)$ conjugating f_h to f . It extends to $W^s(J_\Pi, f_h)$ until it meets a critical point. Each local stable manifold $W_{loc}^s(a)$ ($a \in J_\Pi$) is a complex disk homeomorphic to $\overline{\mathbb{C}} - \overline{\mathbb{D}_R}$ for some $R > 1$. External rays are the rays in $W^s(J_\Pi, f)$ defined by the gradient lines of the Green function G_f restricted to $W_{loc}^s(a)$. Since the Böttcher coordinate transforms the Green function into a canonical form, external rays are the images of the actual rays by the inverse Böttcher coordinate, just as for polynomials of one variable.

Bedford and Jonsson [BJ] established the continuous landing property of external rays for regular polynomial endomorphisms of \mathbb{C}^2 .

Theorem 3.1. *([BJ], Theorem 10.2)*

Let f be a regular polynomial endomorphism of \mathbb{C}^2 . Assume

- (1) f_Π is uniformly expanding on J_Π .
 (2) f is uniformly expanding on $J(f)$.
 (3) The non-wandering set of f in $\partial K(f)$ consists of $J(f)$ and a hyperbolic set \mathcal{S}_1 of unstable index 1.
 (4) $W^s(\mathcal{S}_1) = \bigcup_{\hat{x} \in \hat{\mathcal{S}}_1} W^s(\hat{x})$.
 (5) $W^s(J_\Pi) \cap \mathcal{C}(f) = \emptyset$.
 Then all external rays land onto $J(f)$ and landing points vary continuously.

As a trivial example, we consider the map $F_h(x, y) = (x^2, y^2)$. Then

$$\begin{aligned} \mathcal{C}(F_h) &= \{x = 0\} \cup \{y = 0\}, & K(F_h) &= \{|x| \leq 1, |y| \leq 1\}, \\ J(F_h) &= \{|x| = |y| = 1\}, & F_{h,\Pi}(\zeta) &= \zeta^2, & J_\Pi &= \{|\zeta| = 1\}, \\ W^s(\zeta) &= \{y = \zeta x, |x| > 1\}, & W^s(J_\Pi, F_h) &= \{|x| = |y| > 1\}. \end{aligned}$$

And all the assumptions of the above theorem are satisfied. Then external rays for F_h are labelled by two angles $(\phi, \theta) \in \mathbb{T}^2$. Here $\zeta = y/x = e^{2\pi i \phi} \in J_\Pi$ and θ is the argument of the ray in the disk $W^s(\zeta)$. Hence the external ray $R_{F_h}(\phi, \theta)$ is just the ray :

$$x = re^{2\pi i \theta}, \quad y = \zeta x = re^{2\pi i(\phi + \theta)}, \quad (r > 1),$$

which lands at $(e^{2\pi i \theta}, e^{2\pi i(\phi + \theta)}) \in J(F_h)$.

Our map F is regular but is not expanding on $J(F)$ since $J(F)$ contains critical points, as we will see later. Next lemma says that it satisfies the last condition (5). Its proof also implies that F is critically finite.

Lemma 3.1. $W^s(J_\Pi, F) \cap \mathcal{C}(F) = \emptyset$.

proof. Note that the critical set $\mathcal{C}(F)$ is parametrized as $x = t, y = 1/t$. We calculate the critical orbits and by induction, we show

$$F^n(t, t^{-1}) = (t^{2^n} + 2t^{-2^{n-1}}, t^{-2^n} + 2t^{2^{n-1}}) \quad (n \geq 2).$$

In fact, it is true for $n = 2$. Suppose it is true for $n = k$. Then the first entry of $F^{k+1}(t, t^{-1})$ is

$$\begin{aligned} (t^{2^k} + 2t^{-2^{k-1}})^2 - 2(t^{-2^k} + 2t^{2^{k-1}}) &= t^{2^{k+1}} + 4t^{2^{k-1}} + 4t^{-2^k} - 2t^{-2^k} - 4t^{2^{k-1}} \\ &= t^{2^{k+1}} + 2t^{-2^k}. \end{aligned}$$

The same holds for the second entry. Hence the case $n = k + 1$ is also true.

Note that the map F has two super-attracting fixed points $[1 : 0 : 0]$ and $[0 : 1 : 0]$ in Π and $W^s(J_\Pi, F)$ is contained in the common boundary of their basins. The above

calculation implies that the parts $|t| > 1$ and $|t| < 1$ are contained in the basins of the points $[1 : 0 : 0]$ and $[0 : 1 : 0]$ respectively and the part $|t| = 1$ is contained in $K(F)$. Thus \mathcal{C} never intersects $W^s(J_\Pi, F)$. This completes the proof. \square

Now we consider the external rays for F . Fortunately, we have an explicit expression of an inverse Böttcher coordinate of F and we can define them directly. Put

$$(x, y) = \Psi(u, v) = \left(u + \frac{1}{v} + \frac{v}{u}, v + \frac{1}{u} + \frac{u}{v}\right).$$

Then it satisfies the functional equation

$$F \circ \Psi(u, v) = \Psi(u^2, v^2) = \Psi \circ F_h(u, v).$$

The jacobian $Jac(\Psi)$ is written by

$$Jac(\Psi)(u, v) = \left(1 - \frac{1}{uv}\right)\left(1 - \frac{u}{v^2}\right)\left(1 - \frac{v}{u^2}\right).$$

Hence it is invertible on $W^s(J_\Pi, F_h)$. The inverse $\Phi = \Psi^{-1}$ is a Böttcher coordinate of F .

Then each stable manifold $W_F^s(\zeta)$ of $\zeta \in J_\Pi$ for F is the image of $W_{F_h}^s(\zeta) = \{(t, \zeta t); |t| > 1\} \cong \mathbb{C} - \overline{\mathbb{D}}$ by Ψ . This coordinate gives the Böttcher coordinate of the restriction of F on $W_F^s(\zeta)$. Hence the external ray $R_F(\phi, \theta)$ is the image of $R_{F_h}(\phi, \theta)$ by Ψ .

Theorem 3.2. *The external ray $R_F(\phi, \theta)$ is expressed by*

$$\begin{aligned} x &= re^{2\pi i\theta} + \frac{1}{r}e^{-2\pi i(\phi+\theta)} + e^{2\pi i\phi} \\ y &= re^{2\pi i(\phi+\theta)} + \frac{1}{r}e^{-2\pi i\theta} + e^{-2\pi i\phi} \quad (r > 1). \end{aligned}$$

Its landing point depends continuously on $(\phi, \theta) \in \mathbb{T}^2$:

$$\begin{aligned} x_0 &= e^{2\pi i\theta} + e^{-2\pi i(\phi+\theta)} + e^{2\pi i\phi} \\ y_0 &= e^{2\pi i(\phi+\theta)} + e^{-2\pi i\theta} + e^{-2\pi i\phi} = \overline{x_0}. \end{aligned}$$

Thus $(x_0, y_0) \in H$. Recall that this parametrization of x_0 coincides with that of $K(f)$ described in (2.1) in the previous section.

Lemma 3.2. $K(F) = \{(x, \bar{x}) \in H; x \in K(f)\}$.

proof. Note that the numbers $u, \frac{1}{v}, \frac{v}{u}$ (resp. $v, \frac{1}{u}, \frac{u}{v}$) in the definition of Ψ are the roots of the cubic equation $t^3 - xt^2 + yt - 1 = 0$, (resp. $t^3 - yt^2 + xt - 1 = 0$.) Thus the map $\Psi : (\mathbb{C} - \{0\})^2 \rightarrow \mathbb{C}^2$ is surjective. Hence, for any $(x, y) \in \mathbb{C}^2$, there exists a point

$(u, v) \in (\mathbb{C} - \{0\})^2$ such that $(x, y) = \Psi(u, v)$. Then we have $F^n(x, y) = \Psi \circ F_h^n(u, v)$ and it easily follows that $F^n(x, y) \rightarrow \infty$ if and only if $F_h^n(u, v) \rightarrow \infty$. Since $\Psi(\frac{1}{v}, \frac{1}{u}) = \Psi(u, v)$, it is easy to see that $(x, y) \in K(F)$ if and only if $|u| = |v| = 1$. This completes the proof. \square

Lemma 3.3. $J(F) = K(F)$.

proof. Note that $J(F) \subset K(F)$. Since the critical value set of Ψ intersects $K(F)$ only at the boundary of $K(f)$, Ψ is locally invertible in the interior of $K(F)$ in H . Let Φ_j , $0 \leq j \leq 5$ be the branches of Ψ^{-1} there. Then, we have

$$(dd^c G)^2 = \frac{1}{3} \sum_{j=0}^5 \Phi_j^*(dd^c G_h)^2.$$

Hence, $J(F) = \text{supp}(dd^c G)^2$ contains the image of $J(F_h) = \{|u| = |v| = 1\}$ by Ψ . Thus $K(F) \subset J(F)$. This completes the proof. \square

Now the parameters ϕ and θ turn out to be the external angles for F . Note that $\mathcal{C}(F) \cap H = \{(x, \bar{x}); |x| = 1\}$ coincides with $\mathcal{C}(f)$ and is contained in $J(F)$. See Figure 1. Thus F is not expanding on $J(F)$.

Now Lemma 3.1 says $W^s(J_\Pi) \cap \mathcal{C}(F) = \emptyset$. Then it follows from Theorem 7.4 in [BJ] that Ψ extends to a homeomorphism from $W^s(J_\Pi, F_h)$ onto $W^s(J_\Pi, F)$ conjugating F_h to F . In our case, this is trivial and we have a global parametrization of $W^s(J_\Pi, F)$ as the union of the stable manifolds $W_F^s(\zeta)$ with $\zeta = e^{2\pi i \phi}$.

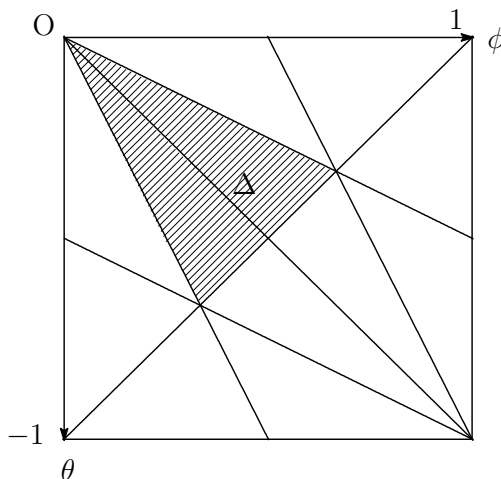


Figure 2: Equivalence on \mathbb{T}^2 and the fundamental region Δ

Note that (ϕ, θ) and the parameters

$$\begin{aligned}\rho_1(\phi, \theta) &= (1 + \theta, \phi - 1) \\ \rho_2(\phi, \theta) &= (\phi, -\phi - \theta) \\ \rho_3(\phi, \theta) &= (-\phi - \theta, \theta)\end{aligned}$$

give a same landing point (x_0, y_0) . That is, several rays land at a same point. We will investigate this in details. We remark that ρ_1 , ρ_2 and ρ_3 are the reflections with respect to the lines $\phi = \theta + 1$, $\phi = -2\theta$ and $\theta = -2\phi$ respectively. These reflections give an equivalence relation in \mathbb{T}^2 . The fundamental region is the closed triangular region Δ surrounded by the three lines :

$$\phi = \theta + 1, \quad \phi = -2\theta, \quad \theta = -2\phi.$$

Figure 2 shows the torus \mathbb{T}^2 , where the dark region indicates the fundamental region Δ . Each triangle is equivalent to one of the two halves of Δ . Now the next lemma is easy to see.

Lemma 3.4. *The equivalence class of a point in the interior of Δ consists of 6 points, while that of a point on one of the three edges of $\partial\Delta$ consists of 3 points and that of a vertex of $\partial J(F)$ consists of a single point itself.*

Since Δ and $\partial\Delta$ correspond respectively to $J(F)$ and $\partial J(F)$, we have the following.

Theorem 3.3. *Each point $z = (x, y)$ in $J(F)$ is the landing point of exactly one, 3 or 6 external rays if z is a cusp point on $\partial J(F)$, z is a non-cusp point on $\partial J(F)$ or $z \in \text{int } J(F)$, respectively.*

Finally note that the restriction of Ψ to H is

$$\Psi(t, \bar{t}) = (t + \frac{1}{\bar{t}} + \frac{\bar{t}}{t}, \bar{t} + \frac{1}{t} + \frac{t}{\bar{t}}) = (\psi(t), \overline{\psi(t)}),$$

where ψ is the Böttcher coordinate of f . Thus the external rays for the map f studied in the previous section are just the restriction of the rays for the map F to H .

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