Kronecker's idea of arithmetization of mathematics

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According to the 19 century German mathematician, Leopold Kronecker, the whole body of mathematics should be constructed rigidly on the basis of intuition of natural numbers. His standpoint is historically called 'intuitionism'. In this article, we examine Kronecker's assertion about the ascendancy of natural numbers over other mathematical entities as the starting building block of mathematics from the perspective of cognitive study of mathematics.

1 Introduction

Rikitaro Fujisawa was one of the first Japanese scholars who studied European-style mathematics. He is also famous for authoring the first series of national arithmetic and algebra textbooks for elementary and middle schools compiled by the Japanese restoration government. In the previous paper [1], we considered how Fujisawa's construction of his pedagogic thoughts—so-called 'enumerationism'—were affected by Leopold Kronecker's lecture in 1887. Although enumerationism textbooks were shortly replaced by the 'green-cover series', his thoughts and methods have had a great influence on the Japanese tradition of mathematics education until today. In the present article we look more closely at Kronecker's own number theory textbook and examine his thoughts on the role of natural numbers in the foundations of mathematics.

2 Background: The discretization program

Perhaps, "discretization" is one of the keywords to understand the history of 19th century European mathematics. G. Lakoff and R. Núñes wrote [2, p.262]:

> In late 19th century Europe, after the success of analytic geometry and calculus, mathematics had gained an important stature as being the discipline that defined the highest form of reason, with precise, rigorous, and indisputable methods of proof. Anything not formalizable was seen as "vague", "intuitive" (as opposed to "rigorous"), and imprecise. At the same time, the foundation of mathematics was seen as crucial to preserving that stature for mathematics. This movement is later called 'the discretization program'.

Kronecker was a 19-century German mathematician, who insisted that the whole body of mathematics should be constructed rigidly on the basis of intuition of natural numbers, and denied the bold arguments of set theory. His standpoint is historically called 'intuitionism'. In intuitionism, both truths and objects of mathematics do not stand independent of mind, but they

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can be grabbed directly by mental activity. Later, Brouwer defined intuitionism more strictly. Brouwer did not allow the use of the law of the excluded middle, and he showed a keen conflict with Dedekind and Cantor.

In what follows, we examine Kronecker's assertion about the ascendancy of natural numbers over other mathematical entities as the starting building block of mathematics from the perspective of cognitive study of mathematics.

3 Kronecker's idea of arithmetization of mathematics

In 1901, Kronecker published his lecture note on number theory [3]. He began this book by giving a brief history of number theory from Babylonian age to 19th century. Although his first concern in this book was number theory, he paid much attention on geometry and analysis in his introductory notes. At the first stage of his lecture, he mentioned that the numbers, especially the whole numbers, were the first mathematical entity of mankind.

Gauss applied analytic methods to number theory and proved the very main theorem of number theory using analysis, and by doing so, Gauss raised a serious problem to the mathematical community in those days about the boundary between number theory and analysis. Kronecker, mentioning Gauss's celebrated results, pointed out that many of the most fundamental constants of geometry and physics had, in their very basic definition, purely arithmetical properties.

For example, Kronecker points out [3, p.4] that the most famous transcendental number π can be defined by the Leibniz series

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2\,n+1},$$

and, he mentions, that this exact (*gerade* in German) representation provides one of the most beautiful arithmetic properties of this vague (*ungerade*) irrational number, which stems from geometry. He writes:

> ... in diesem Sinne ist wohl jenes bekannte Wort: „numero impari deus gaudet“ zu verstehen. (... in this sense, that famous phrase: “the god loves odd numbers” can be understood well [4].)

Indeed, the terms in this series can be distinguished remarkably well arithmetically; the sign of each term varies according to whether the denominator leaves 1 or 3 as residue when divided by 4. Kronecker states that:

> Wir haben hier also eine Definition der Transcendenten π von durchaus zahlentheoretischem Charakter. (We have here, therefore, a definition of the transcendental number π of completely number-theoretical character.)

There are many other formulae including the number π, and some of them have been used to calculate the digits of π, but Leibniz formula is perhaps one of the simplest and most beautiful equations, though it converges very slowly.

Kronecker raised another example, including a continued fraction, which is as simple as the above one.

$$z \cdot \tan z = \cfrac{z^2}{1 - \cfrac{z^2}{3 - \cfrac{z^2}{5 - \cfrac{z^2}{\ddots}}}}$$

The only difference is that it serves as an *implicit* representation of the Rudolf number when one puts $z = \pi/4$ in the both sides. Here again, we see a simple arithmetic sequence of natural numbers [5].

So, what do these examples teach us? These examples are especially beautiful, but Kronecker argues that what we can learn from these examples can be applied universally to all definitions of analysis, i.e., all the definitions appearing in the field of analysis can be reduced to the whole numbers and their properties, and that the whole domain of this branch of mathematics can be explained basically from arithmetic. Therefore, by doing so, we can free analysis from its source domain—geometry. This meant for Kronecker that the boundary of arithmetic could be broadened substantially, and that arithmetic could serve as the foundation of all areas of mathematics, including geometry and physics. Here, Kronecker proposes a new word: "general arithmetic" (*allgemeine Arithmetik*), which includes algebra and analysis.

According to Cantor's set theory, there are much more transcendental numbers than algebraic numbers. On the real line, algebraic numbers are rare compared with other real numbers. But in ordinary mathematics, transcendental numbers we usually treat are not general transcendental numbers, but very special transcendental numbers like π, e, and π^2, etc. Also among rational numbers, we put special emphasis on numbers like 0, 1, and i in mathematics. After all, these numbers are not just numbers like any other numbers. Unlike general rational and irrational numbers, these numbers have conceptual meanings, which is not just the numerical values.

It depends on the idea of what mathematics is, but we can safely say that, for Kronecker, the object of mathematics was only those numbers which can be defined through specific procedures. From the standpoint of concept analysis of mathematics, which is a new branch of cognitive science of today, numbers like π, e, 0, 1, and i have conceptual meaning in a system of common, important, nonmathematical concepts [2, p.450].

4 Ordinal numbers and cardinal numbers

Kronecker's lecture on number theory had some remarkable features. We point out here that there was a strict distinction between cardinal numbers and ordinal numbers in Kronecker's textbook.

The distinction between cardinal numbers and ordinal numbers is generally not mentioned in today's number theory textbooks. The reason is that in today's mathematical context, these concepts are just concerned about the *usage* of numbers in everyday life, and the distinction between them is unified, at the level of mathematics, into the single concept of 'natural numbers', which is the object of mathematics. These concepts of cardinals and ordinals are treated today, not in a course of mathematics, but rather in a course of linguistics.

However, for Kronecker and his contemporary mathematicians, the circumstances were quite different.

After the introductory three lectures of Kronecker's treatment on number theory comes the first part. This part, entitled "Decomposability and congruence in the realm of numbers", is about multiples, divisors, and congruence relations of natural numbers—a standard exposition of elementary number theory. At the very start of this part, Kronecker defines numbers and their operations (addition and multiplication). Let us look at the contents of the fourth lecture i.e. the starting lecture of this part:

> systematic arithmetic, the concept of numbers, the ordinal numbers, the cardinal numbers, the concept of size, addition, commutativity of the addition, the multiplication, commutativity of the factors of a product.

We see from this list that the concept of numbers is immediately followed by the concept of ordinal numbers, and the concept of cardinal numbers comes afterwards. Why? Here we see strict difference between Kronecker's concept of numbers and that of the set theorists.

As a matter of fact, Kronecker did not define numbers. He just stated that *the act of counting* is a natural endowment of human kind. For him numbers are just the *symbols*

$$1, 2, 3, 4, 5, \ldots$$

standing in a line. It was not important what kind of symbols were used; the important point here was the fact that we human beings can immediately understand the structure of the system constructed by these symbols—that these symbols are distinct from each other, that a linear ordering is given *a priori*, with each symbol followed by another symbol, and that the sequence starts with a unique symbol and continues indefinitely. According to Kronecker and other intuitionists, we can perceive what is happening with this linear sequence of infinite symbols at first glance (or hearing), and this ability is innate for all human beings. The symbols and their pronunciation of common use may change from region to region, but the main character of the human mind—the ability to grasp the whole picture of what is going on—is universal throughout mankind.

According to Kronecker, the ability of counting is innate for human beings, and numbers are obtained as a result of the act of counting. We can count just in mind by symbols, but we can also count real things in a line, and that action produces numbers. So, the first kind of numbers we get is the *ordinal number*. After counting a collection of things, the number used for the last thing is determined. This number—the *cardinal number*—describes the 'size' of that collection. By experience, we learn that the cardinal number does not depend on the order we count things in the collection. This is why ordinals come first and cardinals come next.

Once you understood that ordinal numbers and cardinal numbers coincide with each other in counting the items in a collection, addition and multiplication could be defined immediately via making the union and the product set of two sets.

5 Human mind and the construction of mathematics

Perhaps Kronecker's idea about numbers stems from observations of infants' development of number concept. His idea of putting the concept of natural numbers as the basis of mathematics was therefore very natural for the process of human mind.

Mathematics could also be constructed 'rigidly' on the basis of the concept of sets. Then natural numbers can be defined as the size of a finite set. But, as we know from the history of mathematics, set theory caused serious contradictions and controversial philosophical problems. For example, it is very difficult to describe the concept of finiteness of a set within the framework of any axiomatic set theory. In set theory, numbers as sizes of sets (cardinal numbers) are defined, in fact, as *equivalence classes* of the class of (finite) sets. This being too abstract, one has to redefine natural numbers (this time, ordinal numbers) as the terms of an artificial sequence of symbols such as the following:

$$\{\}, \{\{\}\}, \{\{\}, \{\{\}\}\}, \{\{\}, \{\{\}\}, \{\{\}, \{\{\}\}\}\}, \ldots$$

which, starting from the empty set, is so queer that no one can, and no one wants to, manipulate arithmetic using these symbols.

In 1960's and 1970's the world saw an age of modernization of mathematics education. We have learned from these decades that too much dependence on set theory results in inhuman mathematics.

6 Conclusion: Kronecker and mathematics education today

In this article, we have examined Kronecker's idea on the foundation of mathematics. He claimed that the whole region of mathematics— arithmetic, algebra, geometry, analysis, and physics—could be explained rigidly on the basis of arithmetic, and that the system of natural numbers is innate for human mind and can be used as the basis of arithmetic more adequately than sets.

The assertions made by the so-called enumerationist was that the act of counting could also be a good starting point for elementary school mathematics education. Familiarity with natural numbers is the basis of all human mathematical activities. Also, it is a universal fact that infants like to count. According to a Swiss infant psychologist, Jean Piaget (1896-1980), infants begin to count before fully understanding the meaning of numbers. In many traditional cultures, children learn a large amount of various religious and poetic verse by heart, before understanding the real meaning.

R.Fujisawa's series of the first national arithmetic textbooks in Japan begins just by counting. Teachers were encouraged to bring into classrooms many kinds of small things to count, and to practice children to count accurately and fluently. Then it was easy for the children to learn addition of one. Fujisawa recommended that after getting familiar with natural numbers they should learn to count by adding twos like

$$1, 3, 5, 7, 9, 11, 13, 15, \ldots$$

and so forth.

Of course, the whole theory of enumerationism depends substantially on the psychological process of the brain. Fujisawa once said to the audience of nearly 300 elementary and normal school teachers [6], In the future, when psychology is more developed than today, there will surely be new scientific facts, for example, how the children's brain grows, to take into mathematics education.

By the way, according to recent development of cognitive science, human brain is not an all-purpose information processor like the central processing unit of a computer, as was supposed to be in 1960's. Rather, the human mind is 'embodied' in various ways. George Lakoff, one of the main contemporary cognitive scientists, says that one of the various important mechanisms enabling human mind to create, understand, and communicate mathematical ideas is the Basic Metaphor of Infinity (**BMI**). The BMI is a kind of framework to construct or understand various mathematical structures which contain various kinds of infinity. By using the BMI in its various cases, we, human beings with restricted brain and body, can understand mathematical concepts which are constructed by using (the metaphor of) infinite repetition of processes [2].

As for Kronecker, too, the problem around the foundation of mathematics was closely related with philosophy and psychology. That was why he made the famous address "Ueber den Zahlenbegriff" (1887), which was submitted to a bulletin of philosophical lectures, and which he started with the argument about the role of philosophy and his dream of arithmetizing mathematics [7].

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Reference and Comments

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[2] G. Lakoff, R. Núñes, Where Mathematics Comes From, Basic Books, 2000.

[3] Leopold Kronecker, Vorlesungen Über Zahlentheorie, Leipzig, Druck und Verlag von B.G.Teubner, 1901.

[4] The phrase is quoted without mentioning the source. *Cf*. "numero deus impare gaudet"—P.Vergilius Maro (poet, ?-19 B.C.?), Eclogae, 8,75, Serv.

[5] Another important irrational number is e—the base of natural logarithm. This number can be defined by the beautiful formula

$$e = 1 + \frac{1}{1} + \frac{1}{1 \times 2} + \frac{1}{1 \times 2 \times 3} + \frac{1}{1 \times 2 \times 3 \times 4} + \dots,$$

which converges very fast.

[6] 藤澤利喜太郎「数學教授法講義筆記」Tokyo, 1900.

[7] L.Kronecker, Ueber den Zahlenbegriff, Crelle Journal für die reine und angewandte Mathematik. Bd.101. S.337-355. 1887. —K.Hensel ed., Leopold Kroneckers Werke, Teubner, 5 volumes in 6 books, 1895-1931 (Chelsea 1968).