Attractive basins of parabolic fixed points for holomorphic maps of two variables

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In this note, the dynamics of holomorphic maps in $\mathbb{C}^2$ which are tangent to the identity at the origin is investigated. The origin is a parabolic fixed point. A condition to assure the existence of attractive basins is examined.

1 Introduction

In this note, we shall investigate the dynamics of holomorphic maps in $\mathbb{C}^2$ tangent to the identity. That is, we consider the maps of the form:

$$F(x,y) = (f_1(x,y), f_2(x,y)) : \mathbb{C}^2 \to \mathbb{C}^2,$$

where

$$\begin{aligned} f_1(x,y) &= x + p_2(x,y) + p_3(x,y) + \cdots \\ f_2(x,y) &= y + q_2(x,y) + q_3(x,y) + \cdots \end{aligned}$$

are homogeneous expansions at the origin. The origin is a fixed point of F and is parabolic since both of the eigenvalues of the jacobian matrix of F at the origin are 1.

In case of one variable, such a point has an open set $\mathcal{B}$ where all orbits converge to the point. We call such an open set an *attractive basin* of the parabolic fixed point. This is not true in case of several variables. To see this, just consider a polynomial automorphism of $\mathbb{C}^2$. Since such a map has constant jacobian, the jacobian must be 1 if it has a parabolic fixed point with both eigenvalues 1. Then the map is volume preserving and cannot have attractive basins.

So it is an interesting question when an attractive basin exists for a parabolic fixed point and there are several works. Weickert [W] has first shown the existence of attractive basins for an automorphism of $\mathbb{C}^2$ of the form F and investigated the global dynamics in

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Received Aug. 28, 2003

the basin. He used the argument of Ueda [U], who investigated the semi-attractive case, that is, eigenvalues are 1 and b with $|b| < 1$. Hakim [H] generalized the work of Weickert to a wider class of maps. See also Abate [A].

To state their results, we need some notations. We assume the quadratic part $F_2(x, y) = (p_2(x, y), q_2(x, y))$ does not identically vanish. A *characteristic direction* is a direction $v \in \mathbb{C}^2 - \{(0, 0)\}$ such that $F_2(v) = \lambda v$ for some $\lambda \in \mathbb{C}$. It is *non-degenerate* if $F_2(v) \neq (0, 0)$ and *degenerate* otherwise.

Note that a non-degenerate characteristic direction $[v]$ is exactly a fixed point of the rational map $R([x : y]) = [p_2(x, y) : q_2(x, y)]$ in $\mathbb{P}^1(\mathbb{C})$. The *residual index* $\iota(R, [v])$ is defined as the residue fixed point index (cf. Milnor [M]) of the map R at its fixed point $[v]$. A degenerate characteristic direction is a point of indeterminacy of R.

These notions do not depend on the choice of local coordinates. So we may take $v = (1, u)$. Put $r(u) = \dfrac{q_2(1, u)}{p_2(1, u)}$. Then $v = (1, u_0)$ is a characteristic direction if and only if u_0 is a root of $r(u) = u$, that is, a fixed point of r, and the residual index is calculated as follows :

$$\iota(r, u_0) = \frac{1}{2\pi i} \int_{|u-u_0|=\epsilon} \frac{du}{r(u) - u}.$$

If $r'(u_0) \neq 1$, then $\iota(r, u_0) = \dfrac{1}{r'(u_0) - 1}$.

An orbit (x_n, y_n) converges to the origin along the direction v if

$$\lim_{n\to\infty} [x_n : y_n] = [v] \quad \text{in} \quad \mathbb{P}^1(\mathbb{C}).$$

Now we can state the result of Hakim.

Theorem 1.1. *(Hakim [H]) Suppose v is a non-degenerate characteristic direction for F and* $\operatorname{Re} \iota(R, [v]) > 0$. *Then there exists an attractive basin $\mathcal{B}$ where all orbits converge to the origin along the direction v.*

We will examine the necessity of the assumption on the residual index of v.

2 A family of maps

Consider the following maps in $\mathbb{C}^2$.

$$F_c(x, y) = (x + x^2, y + cxy), \quad c \in \mathbb{C}.$$

Note that F_c with $c = 2$ is just the 2-jet of the maps studied in [W].

Since $p_2(x,y) = x^2$, $q_2(x,y) = cxy$, its characteristic directions are exactly $(1,0)$ and $(0,1)$. The former is non-degenerate while the latter is degenerate. We calculate the index of $(1,0)$. Since

$$r(u) = \frac{q_2(1,u)}{p_2(1,u)} = cu,$$

$\iota(r,0) = \dfrac{1}{c-1}$ if $c \neq 1$. Then Theorem 1.1 says that an attractive basin of the origin exists if $\mathrm{Re}\, c > 1$ and that all orbits in this basin tend to the origin along the direction $(1,0)$.

We can completely determine when a basin exists for the map F_c. Put $p(x) = x + x^2$.

Theorem 2.1. *The origin has an attractive basin $\mathcal{B}$ if and only if* $\mathrm{Re}\, c > 0$. *$\mathcal{B}$ is equal to $\mathcal{B}(p) \times \mathbb{C}$, where $\mathcal{B}(p)$ denotes the attractive basin of the parabolic fixed point* 0 *of p. All orbits in $\mathcal{B}$ converge to the origin along the direction* $(1,0)$ *or* $(0,1)$, *if* $\mathrm{Re}\, c > 1$ *or* $0 < \mathrm{Re}\, c < 1$, *respectively.*

Hence, the assumption of Theorem 1.1 is necessary if we take the direction into account. But attractive basins still exist even in a weaker assumption.

We also note that Abate [A] has shown the existence of a "stable manifold" without any assumptions on characteristic directions assuming that the origin is isolated in the fixed point set of F. In our case, the fixed point set of F_c is the y-axis and the origin is not isolated.

Note that the orbit $(x_n, y_n) = F_c^n(x,y)$ of (x,y) is expressed by

$$\begin{aligned} x_n &= p^n(x), \\ y_n &= y \prod_{k=0}^{n-1} (1 + c x_k). \end{aligned}$$

First we consider the dynamics of p in the x-plane. It has a parabolic fixed point at 0. The interior of the filled-in Julia set $K(p)$ of p is equal to the basin $\mathcal{B}(p)$ of 0. But we need the asymptotic behaviour of the orbit x_n in order to study the behaviour of y_n.

Lemma 2.1. *For $x \in \mathcal{B}(p)$,* $x_n = -\dfrac{1}{n} + O(\dfrac{\log n}{n^2})$.

proof. Consider the well known Fatou coordinate of 0. By the coordinate change $x \mapsto z = -1/x$, the dynamics of p around 0 is conjugate to the dynamics of the map

$$g(z) = z + 1 + \sum_{k=1}^{\infty} \frac{1}{z^k}$$

around ∞. So any orbit in the basin of ∞ eventually enters the region Re $z > C_0$ for some large C_0. Since Re $g(z) >$ Re $z + 1/2$ in this region, we may assume Re $g^n(z) > n/2$. Put $\phi_n(z) = g^n(z) - n - \log n$ for $n \geq 1$ and $\phi_0(z) = z$. We will show that ϕ_n converges to a map ϕ in the basin. By the form of g above, it follows

$$\begin{aligned}\phi_{n+1}(z) - \phi_n(z) &= g(g^n(z)) - g^n(z) - 1 - \log \frac{1+n}{n} \\ &= \sum_{k=1}^{\infty} \frac{1}{(g^n(z))^k} - \log(1 + \frac{1}{n}) \\ &= O(\frac{1}{n}).\end{aligned}$$

Thus there exist M and M' such that for $n \geq 1$ we have

$$|\phi_n(z) - z| \leq \sum_{k=0}^{n-1} |\phi_{k+1}(z) - \phi_k(z)| \leq M' \sum_{k=1}^{n-1} \frac{1}{k} \leq M \log n.$$

This holds also for $n = 0$. Then we can improve the estimate above:

$$\begin{aligned}\phi_{n+1}(z) - \phi_n(z) &= \frac{1}{g^n(z)} - \log(1 + \frac{1}{n}) + O(\frac{1}{(g^n(z))^2}) \\ &= \frac{1}{n + \log n + \phi_n(z)} - \frac{1}{n} + O(\frac{1}{n^2}) \\ &= O(\frac{\log n}{n^2}).\end{aligned}$$

And the limit exists :

$$\phi(z) = \lim_{n\to\infty} \phi_n(z) = z + \sum_{k=0}^{\infty} (\phi_{k+1}(z) - \phi_k(z)).$$

Hence $g^n(z) = n + \log n + \phi_n(z) = n + \log n + O(1)$. By the relation $x_n = -1/g_n(z)$, the assertion easily follows. This completes the proof. □

Now we consider the sequence y_n. Theorem 2.1 follows immediately from the next lemma.

Lemma 2.2. *Suppose $x \in \mathcal{B}(p)$. Then there exist constants $A, B > 0$ such that for any $y \neq 0$,*

$$A|y|n^{-a} \leq |y_n| \leq B|y|n^{-a}, \qquad a = \text{Re } c.$$

proof. If we put $c = a + bi$, we have, by the above lemma 2.1,

$$\begin{aligned} |1 + cx_k| &= |1 - \frac{a+bi}{k} + O(\frac{\log k}{k^2})| \\ &= (1 - \frac{a}{k}) \cdot (1 - \frac{bi}{k} + O(\frac{\log k}{k^2})) \\ &= (1 - \frac{a}{k}) \cdot (1 + O(\frac{\log k}{k^2})). \end{aligned}$$

Note that this is true even if $a = 0$. Thus it follows

$$\prod_{k=0}^{n-1} |1 + cx_k| = |1 + cx| \prod_{k=1}^{n-1} (1 - \frac{a}{k}) \prod_{k=1}^{n-1} (1 + O(\frac{\log k}{k^2})) \asymp \prod_{k=1}^{n-1} (1 - \frac{a}{k}).$$

Since

$$\log \prod_{k=1}^{n-1} (1 - \frac{a}{k}) = \sum_{k=1}^{n-1} \log(1 - \frac{a}{k}) = \sum_{k=1}^{n-1} (-\frac{a}{k} + O(\frac{1}{k^2})) = -a \log n + O(1),$$

we have

$$\prod_{k=1}^{n-1} (1 - \frac{a}{k}) \asymp n^{-a}$$

Now the lemma follows easily. This completes the proof. □

Lemma 2.2 also says that, in case Re $c = 1$, the direction of an orbit to the origin depends on the initial point.

By the same argument, it follows

$$\prod_{k=j+1}^{n-1} (1 - \frac{a}{k}) \asymp (\frac{n}{j})^{-a},$$

which will be used later.

3 A perturbation

In this section, we consider a perturbation of the map F_c in the previous section. That is, consider the map:

$$\tilde{F}_c(x, y) = (x + x^2 + f(x), y + cxy + h(x)),$$

where f and g are holomorphic functions of x around the origin and satisfies $f(x) = O(x^3)$, $h(x) = O(x^2)$. It turns out that the same result holds for this map.

Theorem 3.1. *The origin has an attractive basin $\mathcal{B}$ if and only if* $\mathrm{Re}\, c > 0$. *All orbits in $\mathcal{B}$ converge to the origin along the direction* $(1,0)$ *or* $(0,1)$ *if* $\mathrm{Re}\, c > 1$ *or* $0 < \mathrm{Re}\, c < 1$, *respectively.*

Perhaps the most interesting case is $f(x) = 0$ and $h(x) = x^2$. In this case, there are two characteristic directions $(1, 1/(1-c))$ and $(0,1)$. The former is non-degenerate and its index is $1/(c-1)$, while the latter is degenerate.

We will prove Theorem 3.1 only in case $0 < \mathrm{Re}\, c < 1$. Put $\tilde{p}(x) = x + x^2 + f(x)$. Then the same fact as in Lemma 2.1 holds. The proof is also the same.

Lemma 3.1. *For $x \in \mathcal{B}(p)$, $x_n = -\frac{1}{n} + O(\frac{\log n}{n^2})$.*

We need to estimate y_n. It is expressed by

$$y_n = \sum_{j=0}^{n-1} h(x_j) \prod_{k=j+1}^{n-1} (1 + cx_k) + y \prod_{k=0}^{n-1} (1 + cx_k) \equiv I_1 + I_2.$$

Put $\Omega = \tilde{g}^{-1}(\{z \in \mathbb{C}; \mathrm{Re}\ z > K\})$ for large K, where $\tilde{g}$ corresponds to the map g in Lemma 2.1. Then, $\Omega \subset \mathcal{B}(p)$ and $x_k \asymp -1/(k+K)$ for $x \in \Omega$. We estimate

$$I = \prod_{k=j+1}^{n-1} (1 + O(\frac{\log k}{k^2})).$$

Note that for any small $\epsilon > 0$, there exists $C' > 0$ such that

$$\begin{aligned}
\log I &= \sum_{k=j+1}^{n-1} \log(1 + O(\frac{\log k}{k^2})) \\
&\leq C \sum_{k=j+1}^{n-1} \frac{\log k}{k^2} \leq C' \sum_{k=j+1}^{n-1} \frac{1}{k^{2-\epsilon}} \\
&\leq C' \int_j^n \frac{dx}{x^{2-\epsilon}} = \frac{C'}{1-\epsilon}(\frac{1}{j^{1-\epsilon}} - \frac{1}{n^{1-\epsilon}}) \\
&\leq \frac{C'}{(1-\epsilon)K^{1-\epsilon}}.
\end{aligned}$$

So, if we take K sufficiently large, I is arbitrarily close to 1 and we may assume

$$\prod_{k=j+1}^{n-1} |1 + cx_k| \asymp \prod_{k=j+1}^{n-1} (1 - \frac{a}{k}).$$

Suppose $h(x) = hx^r + O(x^{r+1})$ for some constant $h \neq 0$ and $r \geq 2$. Then since $h(x_j) \asymp (-j)^{-r}h$, it follows

$$\begin{aligned} |I_1| = |\sum_{j=0}^{n-1} h(x_j) \prod_{k=j+1}^{n-1} (1 + cx_k)| &\asymp |h| \sum_{j=1}^{n-1} j^{-r} \prod_{k=j+1}^{n-1} (1 - \frac{a}{k}) \\ &\asymp |h| \sum_{j=1}^{n-1} j^{-r} (\frac{n}{j})^{-a} = |h| n^{-a} \sum_{j=1}^{n-1} j^{a-r} \\ &\asymp n^{-a}. \end{aligned}$$

Here we use the fact $a - r \leq a - 2 < -1$ if $0 < a < 1$. On the other hand, from Lemma 2.2, we have $|I_2| \leq B|y|n^{-a}$. Thus, if we take $|y|$ sufficiently small, $y_n \asymp n^{-a}$ and we finish the proof in case $0 < \mathrm{Re}\, c < 1$.

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