

Dynamics of a family of quadratic maps in \mathbf{C}^2

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In this note, the dynamics of a family of quadratic maps in \mathbf{C}^2 is investigated. Especially, the topology of their filled-in Julia sets is studied. It turns out that the filled-in Julia set is a Cantor set if all the critical points escape to infinity.

1 Introduction

In this note, we shall investigate the dynamics of a family of quadratic maps in \mathbf{C}^2 :

$$F_c(x, y) = (x^2 + cy, y^2 + cx), \quad c \in \mathbf{C}.$$

As usual, we define the *filled-in Julia set* K_c of F_c by the set of points (x, y) whose orbit $\{F_c^n(x, y); n \geq 0\}$ is bounded. Since the Jacobian of F_c is

$$D(F_c) = \det \begin{pmatrix} 2x & c \\ c & 2y \end{pmatrix},$$

the set Ω_c of the critical points of F_c is expressed by $4xy = c^2$. It has a parametrization $x = ct/2$, $y = c/(2t)$, $t \in \mathbf{C}$. Let M be the set of parameters c for which a critical point has bounded orbit. That is, its complement is the set of parameters c for which the orbit of Ω_c escapes to infinity. We will show that M is compact. We will also show that, if $c \in \mathbf{C} - M$, then K_c is totally disconnected. When $c \in \mathbf{R}$, \mathbf{R}^2 is invariant under F_c . In this case, we also consider their dynamics on \mathbf{R}^2 .

2 The Green functions

A main tool to investigate the dynamics of polynomial maps is the *Green functions*. To assure the existence of the Green functions, we use the results of Heinemann [H]. For $z = (x_1, \dots, x_n) \in \mathbf{C}^n$, put $\|z\| = \sqrt{\sum_{j=1}^n x_j^2}$.

Definition 2.1 *An entire map $f : \mathbf{C}^n \rightarrow \mathbf{C}^n$ is called a strict polynomial of degree p if there exist constants $k_1, k_2, r > 0$ such that*

$$k_1 \|z\|^p \leq \|f(z)\| \leq k_2 \|z\|^p, \quad \|z\| > r.$$

Lemma 2.1 *F_c is a strict polynomial of degree two.*

proof. If $|x| \sim |y| > r$, it easily follows $\|F_c(x, y)\| \sim |x|^2 + |y|^2 \sim \|(x, y)\|^2$. If $|y| < \epsilon|x|$, then $\|(x, y)\| \sim |x|$ and, since $|x^2 + cy| \sim |x|^2$, $|y^2 + cx| \leq K|x|^2$, we have

$$\begin{aligned} \|F_c(x, y)\| &\leq 2K|x|^2 \sim \|(x, y)\|^2 \\ \|F_c(x, y)\| &\geq |x^2 + cy| \sim |x|^2 \sim \|(x, y)\|^2. \end{aligned}$$

The same holds for $|x| < \epsilon|y|$. This completes the proof. \square

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Lemma 2.2 ([H]) *A strict polynomial $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$ of degree p is proper, onto and has mapping degree p^n . Hence it has p^{nk} periodic points with period k . Especially its filled-in Julia set $K(f)$ is not empty.*

For a strict polynomial f of degree d , we can define the *Green function*. That is, the limit

$$G_f(z) = \lim_{n \rightarrow \infty} \frac{1}{d^n} \log_+ \|f^n(z)\|$$

exists. In fact, for escaping point z , it can be written by

$$G_f(z) = \sum_{k=1}^{\infty} \frac{1}{d^k} \log_+ \frac{\|f^k(z)\|}{\|f^{k-1}(z)\|^d} + \log_+ \|z\|.$$

Since f is a strict polynomial, there exists a $K > 0$ such that

$$\left| \log_+ \frac{\|f^k(x, y)\|}{\|f^{k-1}(x, y)\|^d} \right| \leq K$$

holds for large k . Thus the above series is convergent. Moreover, it follows $G_f(f(z)) = dG_f(z)$ and $K(f) = \{G_f(z) = 0\}$. We denote by $G_c(x, y)$ the Green function of F_c .

3 Symmetry in our family

Our family has some symmetry properties. First let $I(x, y) = (y, x)$. Then it follows $F_c \circ I = I \circ F_c$. Next, put $\omega = (-1 + \sqrt{3}i)/2$ and $J(x, y) = (\omega x, \bar{\omega}y)$. Then, by $\omega^3 = 1$ we have

$$\begin{aligned} F_c(J(x, y)) &= (\omega^2 x^2 + c\bar{\omega}y, \bar{\omega}^2 y^2 + c\omega x) \\ &= (\bar{\omega}(x^2 + cy), \omega(y^2 + cx)) \\ &= J^{-1}(F_c(x, y)), \end{aligned}$$

and consequently, $F_c^2 \circ J = J \circ F_c^2$. Note that J satisfies $J^3 = id$.

Lemma 3.1 *If $z_0 = (x_0, y_0)$ is a k -periodic point of F_c , so is $I(z_0)$. If, in addition, k is even, so are $J(z_0)$ and $J^2(z_0)$ and their eigenvalues are the same. If k is odd, $J(z_0)$ and $J^2(z_0)$ are $2k$ -periodic points and their eigenvalues are squares of those of z_0 .*

It easily follows from the definition that $J \circ I \circ J = I$. Hence, on the diagonal line D , we have $J \circ I \circ J(x, x) = (x, x)$. Then, if (x_0, x_0) is a periodic point of odd period k ,

$$F_c^k(J(x_0, x_0)) = J^{-1}(F_c^k(x_0, x_0)) = J^{-1}(x_0, x_0) = I(J(x_0, x_0)),$$

and we conclude that $(X_0, Y_0) = J(x_0, x_0)$ is a $2k$ -periodic point satisfying $F_c^k(X_0, Y_0) = I(X_0, Y_0)$.

The diagonal line D is F_c -invariant and the dynamics of F_c on D coincides with that of the quadratic polynomial $p_c(x) = x^2 + cx$.

If c is real, the anti-diagonal line $AD = \{y = \bar{x}\}$ is F_c -invariant and its dynamics coincides with that of $f_c(x) = x^2 + c\bar{x}$, which is investigated by Lopes [L1, L2], Alexander et al. [AYZK], and Uchimura [U1, U2, U3].

As an application, we have the following. For $|c| < 1$, the origin $(0, 0)$ is an attracting fixed point of F_c . Let $A_c(0)$ be its *attracting basin* and let $A_c^*(0)$ be its *immediate basin*, i.e. the connected component of $A_c(0)$ containing $(0, 0)$.

Lemma 3.2 *If $|c| < 1$, $A_c(0)$ is connected, that is, $A_c^*(0) = A_c(0)$.*

proof. Since it is trivial for $c = 0$, we may assume $0 < |c| < 1$. We calculate the elements $(x, y) \in F_c^{-1}(0, 0)$. They satisfy the equations $x^2 + cy = 0$, $y^2 + cx = 0$. Solving these, we get $(x, y) = (0, 0)$, $(-c, -c)$, $(-\omega c, -\bar{\omega}c) = J(-c, -c)$, $(-\bar{\omega}c, -\omega c) = J^2(-c, -c)$. Note that $A_c^*(0) \cap D$ corresponds to the immediate basin of the attracting fixed point 0 of p_c . It is well known that this immediate basin coincides with the whole basin, consequently it contains $-c$, a preimage of 0. Hence we have $(-c, -c) \in A_c^*(0)$. By Lemma 3.1, $A_c^*(0)$ is invariant under J . Thus we have $J(-c, -c), J^2(-c, -c) \in A_c^*(0)$. Now suppose $F_c^{-1}(A_c^*(0))$ has a connected component U_c other than $A_c^*(0)$. Since $F_c : U_c \rightarrow A_c^*(0)$ is proper, U_c must contain an element of $F_c^{-1}(0, 0)$. This is a contradiction. Thus it follows $F_c^{-1}(A_c^*(0)) = A_c^*(0)$ and $A_c(0) = \bigcup_{k=0}^{\infty} F_c^{-k}(A_c^*(0)) = A_c^*(0)$. This completes the proof. \square

4 Filled-in Julia sets

In this section, we investigate the topology of filled-in Julia sets. First we give a criterion for a point to escape to infinity. Let $A_c(\infty)$ be the complement of K_c , i.e. the set of all escaping points. Put $F_c^n(x, y) = u_n + iv_n$ for $n \geq 1$ and especially $F_c(x, y) = u + iv$.

Lemma 4.1 *Let K be a constant satisfying $K \geq \frac{2|c| - 1 + \sqrt{8|c|^2 + 1}}{2|c|}$. Then (x, y) escapes if $|x| > 2|c|K + 1$ or $|y| > 2|c|K + 1$. In other words, $K_c \subset \{|x|, |y| \leq 2|c|K + 1\}$.*

proof. Let $K > 1$ be a constant. Since $|u| = |x^2 + cy| \geq |x|^2 - |cy| > (|x| - |c|/K)|x|$ holds for $|x| > K|y|$, it follows $|u| > |x|$ if $|x| > |c|/K + 1$ and $|x| > K|y|$. Moreover, if $|x| > 2|c|K + 1$, we have $|u| > (|c|K + 1)|x|$ and

$$\left|\frac{u}{v}\right| = \left|\frac{x^2 + cy}{y^2 + cx}\right| = \left|\frac{1 + (c/x) \cdot (y/x)}{(y/x)^2 + c/x}\right| > \frac{1 - |c|/K(2|c|K + 1)}{1/K^2 + |c|/(2|c|K + 1)}.$$

We take K so that it satisfies

$$\frac{1 - |c|/K(2|c|K + 1)}{1/K^2 + |c|/(2|c|K + 1)} \geq K,$$

which is equivalent to

$$K(2|c|K + 1) \geq |c| + 2|c|K + 1 + |c|K^2, \quad \text{i.e.} \quad |c|K^2 \geq (2|c| - 1)K + |c| + 1.$$

Thus, if K satisfies $K \geq \frac{2|c| - 1 + \sqrt{8|c|^2 + 1}}{2|c|}$, it follows $|u| > K|v|$. That is, the set $\{|x| > K|y|, |x| > 2|c|K + 1\}$ is F_c -invariant and since $|u_{n+1}| > (|c|K + 1)|u_n|$, it is contained in $A_c(\infty)$. By the same argument, the set $\{|y| > K|x|, |y| > 2|c|K + 1\}$ is F_c -invariant and is contained in $A_c(\infty)$.

Next, consider the case $|x|/K \leq |y| \leq K|x|$. Since

$$|u| \geq |x|^2 - |c||y| \geq (|x| - |c|K)|x|,$$

$|u| > (|c|K + 1)|x|$ holds if $|x| > 2|c|K + 1$. If, in addition, $|u| > K|v|$, (u, v) , consequently (x, y) escapes by the argument above. If $|v| > K|u|$, $|v| > |u| > |x| > 2|c|K + 1$ holds and it also escapes. Otherwise, i.e. if $|u|/K \leq |v| \leq K|u|$, consider the next iteration. Even if $|u_n|/K \leq |v_n| \leq K|u_n|$ for all n , the fact $|u_{n+1}| > (|c|K + 1)|u_n|$ implies (x, y) escapes. In the same way, if $|y| > 2|c|K + 1$, (x, y) escapes. \square

Since the function $g(t) = \frac{2t - 1 + \sqrt{8t^2 + 1}}{2t}$ satisfies $g'(t) = \frac{1}{2t^2} \left(1 - \frac{1}{\sqrt{8t^2 + 1}}\right) > 0$ for $t > 0$, it is monotone increasing for $t > 0$. Hence it follows $1 < g(t) < 1 + \sqrt{2}$ for $t > 0$ and we can take $K = 1 + \sqrt{2}$.

Corollary 4.1 $K_c \subset \{|x|, |y| \leq 2(1 + \sqrt{2})|c| + 1\}$. Especially K_c is compact.

Lemma 4.2 The basin $A_c(\infty)$ is connected.

proof. The proof is the same as in the one dimensional case. Suppose $A_c(\infty)$ is not connected. Then it has a bounded connected component U . Its boundary is contained in K_c . Since $U \subset A_c(\infty)$, there exists a point $(x, y) \in U$ such that, for some n , one component, say u_n , of $F_c^n(x, y)$ has modulus greater than $2|c|K + 1$. By the maximum principle, there is a point $(x', y') \in \partial U$ such that $|u_n(x', y')| > 2|c|K + 1$. By Lemma 4.1, $(x', y') \in A_c(\infty)$, which contradicts the fact $\partial U \subset K_c$. \square

Using the above results, we estimate the set M . Critical points are parametrized by $x = ct/2$, $y = c/(2t)$, $t \in \mathbb{C}$. We estimate the critical values $F_c(ct/2, c/(2t)) = ((\frac{t^2}{4} + \frac{1}{2t})c^2, (\frac{t}{2} + \frac{1}{4t^2})c^2)$. In order to do so, we have only to estimate $h(t) = |\frac{t^2}{4} + \frac{1}{2t}|^2 + |\frac{t}{2} + \frac{1}{4t^2}|^2$.

Lemma 4.3 $h(t)$ takes minimum $1/8$ at $t = -1, -\omega, -\omega^2$.

proof. We regard $h(t)$ as a function of t and \bar{t} . Then it follows

$$\begin{aligned} h_t &= (\frac{t}{2} - \frac{1}{2t^2})(\frac{\bar{t}^2}{4} + \frac{1}{2\bar{t}}) + (\frac{1}{2} - \frac{1}{2t^3})(\frac{\bar{t}}{2} + \frac{1}{4\bar{t}^2}) \\ &= \frac{t^3 - 1}{2t^3}(t(\frac{\bar{t}^2}{4} + \frac{1}{2\bar{t}}) + \frac{\bar{t}}{2} + \frac{1}{4\bar{t}^2}), \\ h_{\bar{t}} &= \overline{h_t}. \end{aligned}$$

Hence $h_t = h_{\bar{t}} = 0$ implies $t^3 = 1$ i.e. $t = 1, \omega, \omega^2$ or $\bar{t}(t^4 + 2t) + 2t^3 + 1 = 0$. The latter case is equivalent to $|t|^2(t^3 + 2) + 2t^3 + 1 = 0$. Putting $t = x + yi$, its real and imaginary parts are written by

$$\begin{aligned} (x^2 + y^2)(2 + x^3 - 3xy^2) + 2x^3 - 6xy^2 + 1 &= 0 \\ y(3x^2 - y^2)(x^2 + y^2 + 2) &= 0 \end{aligned}$$

respectively. From the imaginary part, it follows $y = 0, \pm\sqrt{3}x$. Putting it to the real part, we get $y = 0, x = -1$ or $x = 1/2, y = \pm\sqrt{3}/2$. Thus we have $t = -1, -\omega, -\omega^2$. Calculating the second derivatives,

$$\begin{aligned} h_{tt}(1) &= h_{\bar{t}\bar{t}}(1) = 9/4, & h_{t\bar{t}}(1) &= 0, \\ h_{tt}(-1) &= h_{\bar{t}\bar{t}}(-1) = -1/4, & h_{t\bar{t}}(-1) &= 2. \end{aligned}$$

Put $t = x + iy$ and if we transform them into x, y variables, we have

$$h_{xx} = h_{tt} + 2h_{t\bar{t}} + h_{\bar{t}\bar{t}}, \quad h_{yy} = -h_{tt} + 2h_{t\bar{t}} - h_{\bar{t}\bar{t}}, \quad h_{xy} = i(h_{tt} - h_{\bar{t}\bar{t}}),$$

and it follows

$$\begin{aligned} h_{xx}(1, 0) &= 9/2, & h_{xy}(1, 0) &= 0, & h_{yy}(1, 0) &= -9/2, \\ h_{xx}(-1, 0) &= 7/2, & h_{xy}(-1, 0) &= 0, & h_{yy}(-1, 0) &= 9/2. \end{aligned}$$

Then $t = 1$ turns out to be a saddle and $t = -1$ gives the minimum. Rotational symmetry Lemma 3.1 settles the other cases. \square

Thus we obtain $\min\{|F_c(x, y)|; (x, y) \in \Omega_c\} = |c|^2/(2\sqrt{2})$. By Corollary 4.1, all the critical values escape if

$$|c|^2/(2\sqrt{2}) > \sqrt{2}(2(1 + \sqrt{2})|c| + 1),$$

which is equivalent to $|c| > 4(1 + \sqrt{2}) + \sqrt{52 + 32\sqrt{2}} \sim 19.5$.

Proposition 4.1 $M \subset \{|c| \leq 20\}$, especially M is compact.

proof. We have shown M is bounded. To show M is closed, take a point c_0 in $\mathbb{C} - M$. Then a critical point of F_{c_0} escapes. For c close to c_0 , nearby critical points escape. Thus M is closed. \square

Now consider the dynamics of F_c in case $c \in \mathbb{C} - M$. Let $\Sigma_4 = \{s = (s_0, s_1, s_2, \dots); 0 \leq s_j \leq 3\}$ be the set of sequences of four symbols $\{0, 1, 2, 3\}$ with the usual topology.

Proposition 4.2 If $|c| > 20$, there exists a homeomorphism $\phi_c : K_c \rightarrow \Sigma_4$ which conjugates F_c to the shift map on Σ_4 . Consequently K_c is a Cantor set.

proof. Put $S_c = \{|x|, |y| \leq 2|c|K + 1\}$. Then, from the argument in the proof of Lemma 4.1, it follows $S_c \subset \subset \text{int } F_c(S_c)$, i.e. $F_c^{-1}(S_c) \subset \subset \text{int } S_c$, and we have $K_c = \bigcap_{n \geq 0} F_c^{-n}(S_c)$. If $|c| > 20$, then $F_c(\Omega_c) \cap S_c = \emptyset$, that is, $F_c^{-1}(S_c) \cap \Omega_c = \emptyset$. If we take an open convex neighborhood S'_c of S_c , on each connected component U of $F_c^{-1}(S'_c)$, $F_c : U \rightarrow S'_c$ is proper and injective, hence is an unbranched covering. Since S'_c is simply connected, by the monodromy principle, $F_c : U \rightarrow S'_c$ is biholomorphic. Since F_c is a four-to-one map, $F_c^{-1}(S_c)$ consists of four components S_j , $0 \leq j \leq 3$. Denote by f_j the branch of $F_c^{-1}|_{S'_c}$ taking values on a neighborhood of S_j . For $s \in \Sigma_4$, put

$$D(s_0, s_1, \dots, s_n) = f_{s_0} \circ f_{s_1} \circ \dots \circ f_{s_n}(S_c) = \{(x, y); F_c^j(x, y) \in D(s_j), 0 \leq n\}.$$

It holds $D(s_0) \supset D(s_0, s_1) \supset D(s_0, s_1, s_2) \supset \dots$ and the limit $\Gamma(s) = \bigcap_{n=0}^{\infty} D(s_0, \dots, s_n)$ exists. Then we have

$$K_c = \bigcup_{s \in \Sigma_4} \Gamma(s).$$

In order to show that each $\Gamma(s)$ consists of a single point, we use the following lemma.

Lemma 4.4 (Morosawa et al. [MNTU], Lemma 6.3.7) Let $K_1 \supset K_2 \supset \dots \supset K_k \supset \dots$ be a decreasing sequence of compact sets in \mathbb{C}^N . If there exist an open connected set V in \mathbb{C}^M , a compact set $L \subset V$ and a family $\Phi_k : V \rightarrow \mathbb{C}^N$ of holomorphic maps satisfying

$$K_k \supset \Phi_k(V), \quad \Phi_k(L) \supset K_{k+1}$$

for any k , then $\bigcap_{k=1}^{\infty} K_k$ consists of a single point.

For each $s \in \Sigma_4$, take a subsequence $n(k)$ such that $s_{n(k)}$ is equal to a same j . Put $K_k = D(s_0, s_1, \dots, s_{n(k)-1})$, $\Phi_k = f_{s_0} \circ f_{s_1} \circ \dots \circ f_{s_{n(k)-1}}$, $V = \text{int } S_c$, $L = f_j(S_c) \subset V$. It easily follows

$$\Phi_k(V) = f_{s_0} \circ f_{s_1} \circ \dots \circ f_{s_{n(k)-1}}(\text{int } S_c) \subset K_k$$

and

$$\begin{aligned} \Phi_k(L) &= f_{s_0} \circ f_{s_1} \circ \dots \circ f_{s_{n(k)-1}}(f_{s_{n(k)}}(S_c)) \\ &\supset f_{s_0} \circ f_{s_1} \circ \dots \circ f_{s_{n(k+1)-1}}(S_c) \\ &= K_{k+1}. \end{aligned}$$

By Lemma 4.4, $\Gamma(s) = \bigcap_{k=1}^{\infty} K_k$ consists of a single point $z(s)$. If we define a map $\psi_c : \Sigma_4 \rightarrow K_c$ by $\psi_c(s) = z(s)$, then ψ_c gives a homeomorphism from Σ_4 onto $K_c = \bigcup_{s \in \Sigma_4} \Gamma(s)$ and $\phi_c = \psi_c^{-1}$ is the desired map conjugating F_c to a shift map on Σ_4 . \square

For general $c \in \mathbb{C} - M$, we use the Green function $G_c(x, y)$ and put $G(c) = \min\{G_c(F_c(x, y)); (x, y) \in \Omega_c\}$. Since $U_c = \{(x, y) : G_c(x, y) < G(c)\}$ does not contain any critical values, on each connected component of $F_c^{-1}(U_c)$, F_c is proper and injective, hence is an unbranched covering and is biholomorphic. Since the orbit of Ω_c escapes and the number of components of $F_c^{-1}(U_c)$ is finite, we can connect all of its components by tubes and get an open connected set W which contains $F_c^{-1}(U_c)$ and never meets the orbit of Ω_c . If we take N so that $F_c^N((\mathbb{C} - W) \cup \Omega_c) \subset \mathbb{C} - W$, $(F_c^N)^{-1}$ has 4^N holomorphic branches f_j , $1 \leq j \leq 4^N$ on W . They satisfy $f_j(W) \subset \subset W$. By the same argument as for Proposition 4.2, we have

Proposition 4.3 If $c \in \mathbb{C} - M$, then K_c is a Cantor set.

5 Real dynamics

When c is real, \mathbf{R}^2 is F_c -invariant and we can consider the dynamics of F_c there. Let $K_c(\mathbf{R}) = K_c \cap \mathbf{R}^2$ be the real filled-in Julia set.

Proposition 5.1 *If $1 < c \leq 4$, then $K_c(\mathbf{R}) = \{-c \leq y = x \leq 0\}$.*

proof. Since $K(p_c) \cap \mathbf{R} = [-c, 0]$ holds for $1 < c \leq 4$, it follows $\{-c \leq y = x \leq 0\} \subset K_c(\mathbf{R})$. We show the inverse inclusion relation.

Put $(x_j, y_j) = F_c^j(x, y)$, $j \geq 1$. Since $x_2 = (x^2 + cy)^2 + c(y^2 + cx) > c^2x$ holds, we get $x_{2k} > c^{2k}x$. Then, by Lemma 4.1, (x, y) escapes if $x > 0$. The same holds if $y > 0$. Thus, if $x_j > 0$ or $y_j > 0$ for some j , (x, y) escapes. In other words, $K_c(\mathbf{R}) \subset \{x_j \leq 0, y_j \leq 0\}$ for any j . Especially $K_c(\mathbf{R}) \subset \{x_1 = x^2 + cy \leq 0, y_1 = y^2 + cx \leq 0\} \subset \{-c \leq x, y \leq 0\}$.

Now suppose $K_c(\mathbf{R}) \neq \ell_c \equiv \{-c \leq y = x \leq 0\}$. Since $K_c(\mathbf{R})$ is compact, there exists a point $z = (x, y) \in \ell_c$ which maximizes the distance from the line $y = x$. By the estimate of $K_c(\mathbf{R})$ from above, it follows that $-c \leq x, y \leq 0$ on $K_c(\mathbf{R})$. Here we consider the coordinate $x = X + Y, y = X - Y$. In this coordinate, the map F_c is expressed by $(X, Y) \rightarrow (U, V) = (X^2 + Y^2 + cX, (2X - c)Y)$. Note that $Y = (x - y)/2$ gives $1/\sqrt{2}$ times the distance of the point z from the line $x = y$. Since $X = (x + y)/2 < 0$ holds on $K_c(\mathbf{R})$, it follows $|V| > c|Y| > |Y|$. That is, $F_c(z) \in K_c(\mathbf{R})$ sits further from the line $y = x$ than z . This contradicts the choice of z . Thus we conclude $K_c(\mathbf{R}) = \ell_c$. \square

This argument works also for $c > 4$, where $K_c(\mathbf{R})$ is a Cantor set in ℓ_c .

Next we consider the periodic points on the diagonal D . Let $(x_j, x_j) \in D$, $0 \leq j \leq k-1$ be a k -cycle of F_c for real c . Then x_j , $0 \leq j \leq k-1$ is a k -cycle of p_c . We calculate its eigenvalues. The Jacobian of F_c^k is as follows :

$$D(F_c^k)(x_0, x_0) = \prod_{j=0}^{k-1} D(F_c)(x_j, x_j) = \prod_{j=0}^{k-1} \det \begin{pmatrix} 2x_j & c \\ c & 2x_j \end{pmatrix}.$$

Lemma 5.1 *Put $\Lambda_n = \prod_{j=1}^n \begin{pmatrix} a_j & c \\ c & a_j \end{pmatrix}$ and let $s_j, 1 \leq j \leq n$ be the elementary symmetric polynomial of a_1, a_2, \dots, a_n of degree j . Then it follows*

$$\begin{aligned} \Lambda_{2k} &= \begin{pmatrix} \sum_{j=0}^k s_{2(k-j)} c^{2j} & \sum_{j=0}^{k-1} s_{2(k-j)-1} c^{2j+1} \\ \sum_{j=0}^{k-1} s_{2(k-j)-1} c^{2j+1} & \sum_{j=0}^k s_{2(k-j)} c^{2j} \end{pmatrix}, \\ \Lambda_{2k+1} &= \begin{pmatrix} \sum_{j=0}^k s_{2(k-j)+1} c^{2j} & \sum_{j=0}^k s_{2(k-j)} c^{2j+1} \\ \sum_{j=0}^k s_{2(k-j)} c^{2j+1} & \sum_{j=0}^k s_{2(k-j)+1} c^{2j} \end{pmatrix}. \end{aligned}$$

Here we put $s_0 = 1$.

Hence the eigenvalues of Λ_{2k} are expressed by

$$\rho_{\pm} = \sum_{j=0}^k s_{2(k-j)} c^{2j} \pm \sum_{j=0}^{k-1} s_{2(k-j)-1} c^{2j+1} = \prod_{j=1}^{2k} (c \pm a_j),$$

and those of Λ_{2k+1} by

$$\rho_{\pm} = \pm \left(\sum_{j=0}^k s_{2(k-j)} c^{2j+1} \pm \sum_{j=0}^k s_{2(k-j)+1} c^{2j} \right) = \pm \prod_{j=1}^{2k+1} (c \pm a_j).$$

Thus we obtain the following.

Lemma 5.2 *The eigenvalues of a k -cycle (x_j, x_j) , $0 \leq j \leq k-1$ of F_c are expressed by*

$$\rho_+ = \prod_{j=0}^{k-1} (c + 2x_j) = (p_c^k)'(x_0), \quad \rho_- = (-1)^k \prod_{j=0}^{k-1} (c - 2x_j).$$

Corresponding eigenvectors are $v_+ = (1, 1)$, $v_- = (1, -1)$.

Proposition 5.2 *Suppose $1 < c \leq 4$ and x_0 is a real k -periodic point of p_c . If x_0 is attracting, then the corresponding k -periodic point (x_0, x_0) of F_c on D must be a saddle. If x_0 is repelling, then so is (x_0, x_0) .*

proof. We only consider the case x_0 is attracting. The other case is the same. The eigenvalues of (x_0, x_0) are calculated by Lemma 5.2. From the assumption, it follows $|\rho_+| = |(p_c^k)'(x_0)| < 1$. On the other hand, since $K(p_c) \cap \mathbf{R} = [-c, 0]$ for $1 < c \leq 4$, we have $x_j = p_c^j(x_0) \in [-c, 0]$. Hence

$$|\rho_-| = \prod_{j=0}^{k-1} (c - 2x_j) > 1.$$

and we conclude that (x_0, x_0) is a saddle. \square

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