

Conjectures on Birational Geometry

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In this article we shall prove the rest part of Iitaka C ([Lit]) and Viehweg C^{++} ([Vw1]) conjecture. We make use of Mochizuki theory of Grothendieck anabelian conjecture([Mch]) to the proof which is purely algebraic.

1 Introduction

In 1970 Iitaka programmed A, B, C, \dots conjectures, studying after Kodaira's works. The conjecture C is often said to be Iitaka sub-addition conjecture for Kodaira dimension which Iitaka defined. This conjecture is solved by Kawamata and Viehweg except that a canonical divisor of the generic fibre cannot remove the stable fixed points. To complete the proof, it has remained to be solved one lemma, which we shall prove purely algebraically in this article. The extension of Grothendieck anabelian conjecture solved by Mochizuki is a key tool. Function fields are equivalent to the birational equivalence classes of complete algebraic varieties defined over the complex number field. This statement gives deformation theory of function fields of varieties of non negative Kodaira dimension. It is a known fact that birational automorphism groups of varieties of non negative Kodaira dimension are group schemes locally of finite type. This fact is a critical point which is different from varieties of uni-ruled varieties. We will study a representation of the absolute Galois groups of function fields of base varieties to birational automorphism groups of non uni-ruled varieties. After base change this representation turns to be trivial. An absolute Galois group of a function field of a total variety becomes a semi-direct product of an absolute Galois group of a function field of the geometric generic fibre by an absolute Galois group of a function field of a base variety if the derivation spaces of function fields splits.

2 Conjectures for Birational Geometry

Conjecture 2.1. ([Vw1]) *Let $f : X \rightarrow S$ be a fibre space with a general fibre of Kodaira dimension ≥ 0 . Assume that $\text{var}(X/S) = \dim S$. Then there exists a number $m > 1$ such that $f_*\omega_{X/S}^{\otimes m}$ is big.*

We refer to the following Viehweg lemma.

Lemma 2.1. (p.76[Vw1]) *The sheaves $f_*\omega_{X/S}^{\otimes \nu}$ are big, whenever $\nu \geq 2$ and $\kappa(\det f_*\omega_{X/S}^{\otimes \mu}) = \dim S$ for some $\mu \geq 1$.*

The converse implication is also valid.

We summarize Kawamata-Viehweg theory very briefly and point out the lemma which should be solved.

Theorem 2.1. [Kawamata-Viehweg] *Let $f : X \rightarrow S$ be a connected projective morphism between non singular varieties and $\dim X/S = n$. Assume that*

$$(a) f_*\Omega_{X/S}^n \neq 0$$

$$(b) \text{one has } \ker(\Phi_s : T_{S,s} \rightarrow \text{Hom}(f_*\Omega_{X/S}^n, R^1 f_*\Omega_{X/S}^{n-1})_s) = 0 \text{ for a general point } s \in S \text{ such that } f \text{ is smooth,}$$

Then $\det f_\omega_{X/S}$ is big. Furthermore $f_*\omega_{X/S}^{\otimes m}$ is big for $m \geq 2$ if it is not zero.*

Lemma 2.2. *Let $X \rightarrow S$ be a connected proper surjective morphism between non singular projective varieties and $\dim X/S = n$. Assume that*

$$(a) \dim \ker(\Phi_s : T_{S,s} \rightarrow \text{Hom}(f_*\Omega_{X/S}^n, R^1 f_*\Omega_{X/S}^{n-1})_s) > 0 \text{ for a general point } s \in S.$$

Then one has $\text{var}(X/S) \leq \dim S - \dim \ker \Phi_s$.

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Proof. Let U be the open subvariety of S such that $f : X \rightarrow S$ is smooth. The variation of Hodge structure defined by $R^n f_* \mathbb{C}$ gives the Griffiths map $\Psi : U \rightarrow D/\Gamma$. The kernel of the differential $d\Psi$ is a direct summand of $\ker \Psi$. Hence $\text{var}(X/S) \leq \dim S - \dim \ker \Phi_s$. This will be proved by the next lemma. \square

The next lemma is a key role for proving the conjecture. One will see the complete proof using the absolute Galois groups due to Mochizuki's theorem later.

Lemma 2.3. *Let $X \rightarrow S$ be a connected proper surjective morphism between non singular projective varieties and $\dim X/S = n$. Assume that*

- (a) $f_* \Omega_{X/S}^n \neq 0$, hence Kodaira dimension of a general fibre of X/S is non negative,
- (b) $\dim \ker(\Phi_s : T_{S,s} \rightarrow \text{Hom}(f_* \Omega_{X/S,s}^n, R^1 f_* \Omega_{X/S,s}^{n-1})) = \dim S$ for a general point $s \in S$.

Then one has $\text{var}(X/S) = 0$.

Proof. Note that $\Omega_{X/S,s}^{n-1} \cong \Omega_{X/S,s}^{n-1} \otimes \Omega_{X/S,s}^n$ for a general $s \in S$. By assumption infinitesimal Hodge bundles are invariable, so is $f_* \Omega_{X/S}^n$. Hence Kodaira-Spencer type map $\rho_s : T_{S,s} \rightarrow R^1 f_* \Theta_{X/S}(K_{X/S})_s$ is a zero map. Hence one has the following exact sequence; $0 \rightarrow f_* \Theta_{X/S}(K_{X/S}) \rightarrow f_* \Theta_X(K_{X/S}) \rightarrow T_S \rightarrow 0$ on an open subscheme of S . One obtains the lemma using the argument of the representation of the absolute Galois group of the function field of a variety of Kodaira dimension ≥ 0 . \square

Hence one obtains the statement to be shown in the following section.

Lemma 2.4. *Let $f : X \rightarrow S$ be a proper connected surjective morphism between projective non singular varieties with a general fibre of Kodaira dimension ≥ 0 . Assume that one has the following exact sequence; $0 \rightarrow f_* \Theta_{X/S}(K_{X/S}) \rightarrow f_* \Theta_X(K_{X/S}) \rightarrow T_S \rightarrow 0$ on an open subscheme of S . Then $\text{var}(X/S) = 0$.*

Remark 2.1. *If $\text{var}(X/S) = \dim S$, $f_* \omega_{X/S} = 0$ and the general fibre of X/S is of Kodaira dimension ≥ 0 , take a resolution $\delta : X' \rightarrow X$ of the cyclic covering defined by $(\omega_{X/S} \otimes H)^{\otimes N} = \mathcal{O}_X(D)$ where H is an ample invertible sheaf S and D is a normal crossing divisor on X , if necessary, X replaced by a blow-ups of X . Since $\text{var}(X'/S) \geq \text{var}(X/S)$, one has $\text{var}(X'/S) = \dim X/S$. Hence Kawamata-Viehweg theorem implies that $f'_* \omega_{X'/S}^{\otimes \nu}$ is big for $\nu \geq 2$. Hence there exists a number $\mu \geq 2$ such that $f_* \omega_{X/S}^{\otimes \mu}$ is big. Thus $\kappa(\det f_* \omega_{X/S}^{\otimes m}) = \dim S$ for some $m \geq 1$.*

3 Profinite group theory(semidirect product, direct product, central product)

We make use of the following theorem by Mochizuki.

Theorem 3.1. *Let p be a prime number. Let k be a subfield of a finitely generated field extension of \mathbb{Q}_p . Let L, M be function fields of arbitrary dimension over K . Let $\text{Hom}_{\text{Spec}(k)}(\text{Spec}(L), \text{Spec}(M))$ be the set of k -morphisms from $\text{Spec}(L)$ to $\text{Spec}(M)$. Let $\text{Hom}_{\Gamma_k}^{\text{open}}(\Gamma_L, \Gamma_M)$ be the set of open continuous group homomorphisms $\Gamma_L \rightarrow \Gamma_M$ over Γ_k , considered up to composition with an inner automorphism arising from $\ker(\Gamma_M, \Gamma_k)$. Then the natural map $\text{Hom}_{\text{Spec}(k)}(\text{Spec}(L), \text{Spec}(M)) \rightarrow \text{Hom}_{\Gamma_k}^{\text{open}}(\Gamma_L, \Gamma_M)$ is bijective.*

We provide some preliminaries

Lemma 3.1. *Let $1 \rightarrow H \rightarrow G \rightarrow K \rightarrow 1$ be an exact sequence of groups. Then one has a representation $\rho : K \rightarrow \text{Out}(H)$ and $\rho(K) = G/H C_G(H)$.*

Proof. For $g \in N_G(H)$ the map $h \in H \rightarrow ghg^{-1} \in H$ is an element of $\text{Aut}(H)$. Thus $H \cap N_G(H)/(H \cap C_G(H)) \subset N_G(H)/C_G(H) \subset \text{Aut}(H)$. Since $\text{Inn}(H) \cong H/C(H) \cong HC_G(H)/C_G(H)$, one has

$$N_G(H)/HC_G(H) \subset \text{Out}(H).$$

Thus one has

$$\rho : K = G/H \rightarrow G/H C_G(H) \subset \text{Out}(H).$$

Since $K = G/H$, $K \rightarrow G/H C_G(H)$ is surjective. Hence $\rho(K) = G/H C_G(H)$. \square

Corollary 3.1. *Let $1 \rightarrow H \rightarrow G \rightarrow K \rightarrow 1$ be an exact sequence of groups. If $\rho = 1$, $G = HC_G(H)$ is a central product, i.e., $1 \rightarrow C(G) \rightarrow H \times C_G(H) \rightarrow G \rightarrow 1$ is an exact sequence. Here $C(G) \cong H \cap C_G(H) = C(H)$.*

Proof. The representation $\rho : K \rightarrow \text{Out}(H)$ factors through an injection $G/\text{HC}_G(H) \hookrightarrow \text{Out}(H)$. H and $C_G(H)$ commute elementwise. \square

Lemma 3.2. *Let H be a normal subgroup of G and K a subgroup of G . Assume that*

- (a) $G = HK$ is a semidirect product.
- (a') $1 \rightarrow H \rightarrow G \rightarrow K \rightarrow 1$ is an exact sequence and it splits.
- (c) $C(G) = 1$ or $C(H) = 1$

Then $G = \text{HC}_G(H)$ is a direct product. If $\rho = 1$, $G = HK$ is a direct product, i.e., $K = C_G(H)$.

Proof. One has an exact sequence $1 \rightarrow H \rightarrow G \rightarrow K \rightarrow 1$. Since $K = G/H$, $K \rightarrow G/\text{HC}_G(H)$ is surjective. Hence $\rho(K) = G/\text{HC}_G(H)$. If $\rho = 1$, $G = HK = \text{HC}_G(H)$. When $C(G) = 1$ or $C(H) = 1$, $G = \text{HC}_G(H)$ is a direct product. Since one has two exact sequences;

$$1 \rightarrow H \rightarrow G \rightarrow K \rightarrow 1$$

$$1 \rightarrow H \rightarrow G \rightarrow C_G(H) \rightarrow 1,$$

Hence K is isomorphic to $C_G(H)$. Since $G = HK$ is a semidirect product, the composition $K \hookrightarrow G \rightarrow C_G(H)$ is an isomorphism. Hence $K = C_G(H)$ and $G = HK$ is a direct product. \square

Lemma 3.3. *Let $1 \rightarrow H \rightarrow G \rightarrow K \rightarrow 1$ be an exact sequence. Assume*

- (a) $\text{Out}(H)$ is abelian.
- (b) $K = [K, K]$

Then the representation $\rho : K \rightarrow \text{Out}(H)$ is trivial.

Proof. Since $\text{Out}(H)$ is abelian, the representation $\rho : K \rightarrow \text{Out}(H)$ factors through $K \rightarrow K/[K, K]$. Hence $\rho = 1$. \square

Lemma 3.4. *Let $1 \rightarrow H \rightarrow G \rightarrow K \rightarrow 1$ be an exact sequence. Assume*

- (a) $\text{Out}(H)$ is finite.

Then $[K : \ker(\rho : K \rightarrow \text{Out}(H))]$ is finite.

Proof. It is obvious to look at $\rho : K \rightarrow \text{Out}(H)$. \square

Definition 3.1. *The pullback of an exact sequence $1 \rightarrow H \rightarrow G \rightarrow K \rightarrow 1$ by a group homomorphism $K' \rightarrow K$ is defined to be $1 \rightarrow H \rightarrow G \times_K K' \rightarrow K' \rightarrow 1$.*

Lemma 3.5. *Let $1 \rightarrow H \rightarrow G \rightarrow K \rightarrow 1$ be an exact sequence. Assume*

- (a) $\text{Out}(H)$ is finite.

Let $K' = \ker(\rho : K \rightarrow \text{Out}(H))$. One has the pullback exact sequence; $1 \rightarrow H \rightarrow G \times_K K' \rightarrow K' \rightarrow 1$ such that $\rho' : K' \rightarrow \text{Out}(H)$ is a trivial representation.

Proof. Obvious. \square

Lemma 3.6. *Let $1 \rightarrow H \rightarrow G \rightarrow K \rightarrow 1$ be an exact sequence. Assume*

- (a) $\text{Out}(H)$ is abelian.
- (b) $[K, K] = DK$ is of finite index in K .

Then putting $K' = [K, K] = DK$ for the pullback exact sequence $1 \rightarrow H \rightarrow G \times_K K' \rightarrow K' \rightarrow 1$. the representation $\rho' : K' \rightarrow \text{Out}(H)$ is trivial.

Proof. Obvious. \square

4 Lie algebras of the derivatives of function fields

Lemma 4.1. *Let k be an algebraically closed field of characteristic 0 and L, M function fields over k . Let $\text{Spec } L \rightarrow \text{Spec } M$ be a dominant separable morphism over $\text{Spec } k$. Assume that M is algebraically closed in L . There exists an exact sequence of Lie algebras of the derivatives which splits;*

$$0 \rightarrow \text{Der}_M L \rightarrow \text{Der}_k L \rightarrow L \otimes \text{Der}_k M \rightarrow 0.$$

L is finitely generated over M : hence $L = M(x_i)$. Assume $D(x_i) = 0$ for a basis $B = (D)_{D \in \text{Der}_k(M)}$ of $L \otimes \text{Der}_k(M)$. Let $E = \cap_{D \in B} \ker(D : L \rightarrow L)$. Then E is a function field.

Proof. $D(m) = 0$ for $D \in \text{Der}_M(L)$ and for $m \in M$.

Put $E = \cap_{D \in B} \ker D$. Then E is a field. In fact, if $x \in L$ is algebraic over E , $D(x) = 0$ for any $D \in B$. Hence E is algebraically closed in L . Let $g(\xi_i)$ be a polynomial function of indeterminates ξ_i such that $g(x_i) \neq 0$. One has $D(\frac{1}{g(x_i)}) = 0$ for $D \in \text{Der}_k(M)$. Obviously $L = M(x_i) = M(E) = E(M)$. M is finitely generated over k . Thus L is finitely generated over E . \square

Lemma 4.2. *Let $X \rightarrow S$ be a connected proper surjective morphism between non singular projective varieties and $\dim X/S = n$. Assume that*

- (a) $f_* \Omega_{X/S}^n \neq 0$, hence Kodaira dimension of a general fibre of X/S is non negative,
- (b) $0 \rightarrow f_* \Theta_{X/S}(K_{X/S}) \rightarrow f_* \Theta_X(K_{X/S}) \rightarrow T_S \rightarrow 0$ on an open subscheme of S .
- (c) S is a curve.

Let $\mathcal{O}_F = \ker(D : \mathcal{O}_X \rightarrow \mathcal{O}_X)$ for a $D \in f_ \Theta_X(K_{X/S})$ lifted from a non zero vector field of T_S over an open subscheme of S . Let E be a function field defined by \mathcal{O}_F . Then one has $R(X) = R(S)(E)$.*

Proof. Let U be an affine open of X . The natural morphism $p : \text{Spec } \mathcal{O}_X(U) \rightarrow \text{Spec } \mathcal{O}_\Gamma(U)$ is dominant. Let E be the total quotient field of $\mathcal{O}_F(U)$. Then $\text{Spec } R(S) \otimes E$ corresponds to the generic fibre of X/S . Hence $R(X) = R(S)(E)$. \square

Note that by Mochizuki's correspondence, one has continuous open group homomorphisms; $\Gamma_L \rightarrow \Gamma_M$ and $\Gamma_L \rightarrow \Gamma_E$ such that

$$\begin{array}{ccc} \Gamma_L & \longrightarrow & \Gamma_E \\ \downarrow & & \downarrow \\ \Gamma_M & \longrightarrow & \Gamma_k \end{array} \quad (4.1)$$

$\Gamma_L = \Gamma_E \Gamma_M$ is a semidirect product.

Proposition 4.1. *Let k be an algebraically closed field of characteristic 0 and L, M function fields over k . Let $\text{Spec } L \rightarrow \text{Spec } M$ be a dominant separable morphism over $\text{Spec } k$. Assume that M is algebraically closed in L and $\dim_k M = 1$. One has*

$$\text{Der}_k L = L \otimes \text{Der}_k M \oplus \text{Der}_M L.$$

E is a function field defined by $\ker(D : L \rightarrow L)$ for a $D \in L \otimes \text{Der}_k M$. Assume that $L = M(E)$. Then $\Gamma_L = \Gamma_E \Gamma_M$ is a semidirect product.

Lemma 4.3. *Let $f : G \rightarrow K$ be a continuous homomorphism between profinite groups. Then $\ker f$ is a closed normal subgroup of G .*

Proof. $f^{-1}(1)$ is a closed normal subgroup since 1 is closed \square

Note that a subgroup of a profinite group is profinite. [p.28 Prop.2.2.1, RBZL]

Definition 4.1. *Let G be a profinite group and denote by $\text{Aut}(G)$ the group of all continuous automorphisms of G . For a closed normal subgroup K of G , define*

$$A_G(K) = \{\phi \in \text{Aut}(G) | \phi(g)g^{-1} \in K \text{ for all } g \in G\}$$

Definition 4.2. *Let G be a profinite group and H a subgroup of G . One denotes by $H \triangleleft_c G$ H a closed normal subgroup of G .*

Furthermore, the canonical representation $\rho : G/H \rightarrow \text{Out}(H)$ is continuous.

Note that there exist pullbacks and pushouts in the category of profinite groups.

Let $1 \rightarrow H \rightarrow G \rightarrow K \rightarrow 1$ be an exact sequence of profinite groups. Take a pullback by $D(K) \rightarrow K$.

$$\begin{array}{ccc} G \times_K D(K) & \longrightarrow & D(K) \\ \downarrow & & \downarrow \\ G & \longrightarrow & K \end{array} \quad (4.2)$$

One has an exact sequence $1 \rightarrow H \rightarrow G \times_K D(K) \rightarrow D(K) \rightarrow 1$.

Lemma 4.4. *Let H be a profinite group. On $\text{Aut}(H)$ the congruence subgroup coincides with the compact-open topology.*

Note that the neutral component of an algebraic group of the birational automorphism group of a complete non singular variety of Kodaira dimension ≥ 0 is an abelian variety. (Matumura) Accademia nazionale dei lincei Estratto dai Rendiconti della Classe di Scienze fisiche, matematiche e naturali Setie VII, vol. XXXIV, fasc. 2.-Febbraio 1963 Geometria algebrica. - On Algebraic Groups of Birational Transformations. Nota di Hideyuki Matsumura, presentata dal Socio B. Segre. The cross section theorem of M. Rosenlicht Some basic theorems on algebraic groups, Amer. J. Math. 78(1956)

Theorem 4.1. *Assume that a connected linear group H operates on V non trivially and let B be a Borel subgroup of H . Then V is birationally equivalent to $\mathbb{P}^s \times V_B$ where V_B is the variety of B -orbits of V and $0 < s \leq \dim B$.*

Theorem 4.2. *A necessary and sufficient condition that $\text{Bir}(V)$ contains a linear algebraic group of positive dimension is that V is birationally equivalent to the product of \mathbb{P}^1 and another variety. In particular, if a complete non singular variety V has a multicanonical system $|nK|$ for some $n > 0$, then $\text{Bir}(V)$ cannot contain any linear algebraic group of positive dimension.*

Note that the birational automorphism group of a complete non singular variety of Kodaira dimension ≥ 0 is a group scheme locally of finite type with countable components. (Iitaka)

Lemma 4.5. *Let G be a profinite group.*

(a) *Let C_1, C_2, \dots be a countably infinite set of nonempty closed subsets of G having empty interior. Then*

$$G \neq \bigcup_{n=1}^{\infty} C_n.$$

(b) *The cardinality $|G|$ of G is either finite or uncountable.*

Proposition 4.2. *Every quotient group G/K of a profinite group G , where $K \triangleleft_c G$, is a profinite group.*

Lemma 4.6. *Let $\rho : K \rightarrow F$ be a homomorphism of groups. One has an induced homomorphism $\rho : K/D(K) \rightarrow F/D(F)$ and $\rho(D(K)) \subset D(F)$.*

Lemma 4.7. *Let G be a profinite group and H a normal closed subgroup. The congruence subgroup topology on $\text{Aut}(H)$ is the coarsest among the topologies such that $H \times \text{Aut}(H) \rightarrow H$ is continuous.*

Let G, K be profinite groups and $\phi : G \rightarrow K$ a continuous homomorphism of profinite groups. Let H be $\ker(\phi : G \rightarrow K)$. $\{U\}$ are open normal subgroups on H and form a system of fundamental neighbourhoods of 1. This is an induced topology on a subgroup H . $\{V\}$ are open normal subgroups on G and form a system of fundamental neighbourhoods of 1. Thus $U = V \cap H$. One denotes the group of continuous automorphisms of H by $\text{Aut}(H)$. One endows $\text{Aut}(H)$ a system of fundamental neighbourhoods $A_H(U)$ whose automorphism leaves U invariant and induced action on H/U is trivial. This topology is said to be the congruence subgroup topology or the compact-open topology.

Remark 4.1. *The congruence subgroup topology is Hausdorff, i.e., $\bigcap_U A_H(U) = 1$.*

For every $g \in G$ an automorphism of H defined by $h \rightarrow ghg^{-1}$ for $h \in H$ gives a representation $\rho : G \rightarrow \text{Aut}(H)$. One has $\ker(\rho : G \rightarrow \text{Aut}(H)) = C_G(H)$.

Lemma 4.8. *The representation $\rho : G \rightarrow \text{Aut}(H)$ of profinite groups is continuous.*

Proof. The canonical quotient $G \rightarrow G/V$ induces $H \rightarrow HV/V$ and an isomorphism $H/U = H/H \cap V \cong HV/V$. For every $g \in V$ the automorphism of HV/V defined by $hv \rightarrow ghvg^{-1} = ghg^{-1}gvg^{-1}$ is trivial since $1VhV(1V)^{-1} = hV$ for any $h \in H$ using $hVh^{-1} = V, V^{-1} = V$. The action of $g \in V$ as an element of $Aut(H)$ upon H/U is trivial and its action on U is invariant. Hence $\rho^{-1}A_G(U)$ contains V . Thus $\rho : G \rightarrow Aut(H)$ is continuous. \square

Lemma 4.9. *One has the representation $G/H \rightarrow Out(H)$. This representation is continuous.*

Proof. $Aut(H)/Inn(H) = Out(H)$. The inner automorphism group $Inn(H)$ is the image of the canonical map $H \rightarrow Aut(H)$. \square

Remark 4.2. *Let $\phi : G \rightarrow K$ be a continuous epimorphism of profinite groups. Then $H = \ker \phi$ is closed.*

The epimorphism of profinite groups $G \rightarrow K$ is a strict homomorphism since G is compact and K is Hausdorff, i.e., $G/H \cong K$ is an isomorphism of topological groups or bicontinuous. [TGIII.16 Remarques 1)]

Theorem 4.3. *Let $\phi : G \rightarrow K$ be an open continuous epimorphism of profinite groups. Let $H = \ker \phi$. Then $H \triangleleft_c G$. The representation $K \rightarrow Out(H)$ is continuous, when $Out(H)$ is endowed with the topology induced by the congruence subgroup topology.*

Proof. Since G acts continuously on H by $g \rightarrow (h \rightarrow ghg^{-1})$, $G \rightarrow Aut(H)$ is continuous when $Aut(H)$ is endowed with the congruence subgroup topology. Thus $K = G/H \rightarrow Aut(H)/Inn(H) = Out(H)$ is continuous. \square

Remark 4.3. *Assume that $Out(H)$ is a group scheme locally of finite type with countable components. Assume that the neutral component is an abelian variety A . Given $Out(H)$ the topology induced by the congruence subgroup topology on $Aut(H)$, the closure of A is a commutative group. Since A is a normal subgroup of $Out(H)$, the closure \bar{A} of A is also a normal closed subgroup of $Out(H)$. Hence $Out(H)/\bar{A}$ is a countable group, which is totally disconnected.*

Let $\phi : G \rightarrow K$ be an open continuous epimorphism of profinite groups and denote the kernel ϕ by H . The group $Out(H)$ is endowed with the quotient topology induced by the congruence subgroup topology. Then $\rho : K \rightarrow Out(H)$ is a continuous representation.

Assume that $Out(H)$ is a group scheme locally of finite type with countable components with respect to Zariski topology and that the neutral component with respect to Zariski topology is an abelian variety A . Forget Zariski Topology and take the closure \bar{A} of A with respect to the quotient topology on $Out(H)$ induced by the congruence subgroup topology $Aut(H)$. Then $\rho^{-1}(\bar{A})$ is a closed normal subgroup.

Lemma 4.10. *$K/\rho^{-1}(\bar{A})$ is a profinite group; hence a finite group.*

Proof. Since $\rho^{-1}(\bar{A}) \triangleleft_c K$, $K/\rho^{-1}(\bar{A})$ is a profinite group. $K/\rho^{-1}(\bar{A}) \rightarrow Out(H)/\bar{A}$ is injective. $Out(H)/\bar{A}$ is countable. Since there is no countable profinite group, $K/\rho^{-1}(\bar{A})$ is a finite group. \square

Pulling back an epimorphism $G \rightarrow K$ of profinite groups by a monomorphism $\rho^{-1}(\bar{A}) \rightarrow K$, one denotes by K' $\rho^{-1}(\bar{A})$ and one obtains a continuous representation $\rho : K' \rightarrow Out(H)$ and an exact sequence

$$1 \rightarrow H \rightarrow G \times_K K' \rightarrow K' \rightarrow 1.$$

The representation $\rho : K' \rightarrow Out(H)$ factors through the normal closed subgroup \bar{A} . Furthermore, the restriction of the representation $\rho : K' \rightarrow Out(H)$ to the closure of the commutator subgroup $D(K') = [K', K']$ is trivial.

Theorem 4.4. *Let $f : X \rightarrow S$ be a projective surjective connected morphism between projective non singular varieties with a fibre of Kodaira dimension ≥ 0 . Assume that the right exact sequence $\Gamma_{R(X)} \rightarrow \Gamma_{R(S)} \rightarrow 1$ splits. Then $var(X/S) = 0$.*

Proof. One denotes the absolute Galois groups by $G = \Gamma_{R(X)}, K = \Gamma_{R(S)}$. One may assume that $\phi : G \rightarrow K$ is an open continuous epimorphism up to inner automorphism of K corresponding to $Spec R(X) \rightarrow Spec R(S)$. One denotes by $H = \ker(\phi : G \rightarrow K)$. By assumption, G is the topological semi-direct product $G = H \rtimes K$. After one takes a finite cover $S' \rightarrow S$ associated to the closed subgroup K' of K and a maximal abelian cover $S'' \rightarrow S'$ associated to the closed subgroup $\bar{D}(K')$ of K' , the representation $\rho : \bar{D}(K') \rightarrow Out(H)$ is trivial. Hence $G'' = H \rtimes \bar{D}(K')$ becomes a direct product. Thus the main component X'' of $X \times_S S''$ is birationally equivalent to a product $F \times S''$ for a projective variety F obtained by taking a projective model for the fixed field of $R(X'')$ by the action of the normal closed subgroup $\bar{D}(K')$ of G'' . \square

5 Log Variety

Let R be a commutative integral ring with unity and p, \dots, q prime ideals of height 1. Put $S = R \setminus (p \cup \dots \cup q)$. Then S is a multiplicative set of R . One denotes by $R_{p, \dots, q} = S^{-1}R$. This is a semi-local ring.

Definition 5.1. Let X be a projective regular variety and D a normal crossing divisor on X . The couple (X, D_X) is said to be a log variety. Let x_i be a generic point of D_i .

Let one denote by $\mathcal{O}_{X, \underline{x}}$ the semi-local ring corresponding to $\underline{x} = (x_i)$.

Proposition 5.1. Let X, Y be projective normal varieties. If X, Y are birational each other, then the sets of points of codimension 1 on X, Y are bijective each other.

Proof. The point of indeterminacy of rational maps is not less than of codimension 2. \square

Definition 5.2. Let $f : X \rightarrow Y$ be a rational map between projective normal varieties. For a point y of codimension 1 of Y one can define the inverse image $f^{-1}(y)$ to be the set of all the points x such that $f(x) = y$.

Proposition 5.2. Let $(X, D_X), (Y, D_Y)$ be two log varieties. If $\mu : Y \setminus D_Y \rightarrow X \setminus D_X$ is a proper birational morphism, there exists points $\underline{y} = (y_i)$ on Y of height 1 corresponding to a subset of generic points of D_Y such that $\mathcal{O}_{X, \underline{x}} \cong \mathcal{O}_{Y, \underline{y}}$.

Proposition 5.3. Let $f : (X, D_X) \rightarrow (Y, D_Y)$ be a surjective morphism of log varieties. Then there exists a unique maximal $D'_X \subset D_X$ such that the induced morphism $f' : (X, D'_X) \rightarrow (Y, D_Y)$ is factored through by f and that D'_X has no vertical component with respect to f' .

Proof. Obvious. \square

Definition 5.3. (Itaka) $f : (X, D_X) \rightarrow (Y, D_Y)$ is said to be a strictly rational map if there exist a proper birational morphism $\mu : (Z, D_Z) \rightarrow (X, D_X)$ and $g : (Z, D_Z) \rightarrow (Y, D_Y)$ such that $f \circ \mu = g$ as rational maps.

Proposition 5.4. The set of dominant strictly rational maps from (X, D_X) to (Y, D_Y) is the set of extensions $\phi : R(Y) \subset R(X)$ such that for any generic point y of $D_{Y, j}$ $f^{-1}(y) \subset \{x_i\}$ where x_i corresponds to the generic point of a component $D_{X, i}$. Here f is defined by ϕ .

Proposition 5.5. Let $f : (X, D_X) \rightarrow (Y, D_Y)$ be a dominant strictly rational map. Then there exists a divisor $D'_X \subset D_X$ such that $\text{Spec } \mathcal{O}_{X, \underline{x}} \rightarrow \text{Spec } \mathcal{O}_{Y, \underline{y}}$ is a surjective morphism. Here $\text{Spec } \mathcal{O}_{X, \underline{x}}$ (resp. $\text{Spec } \mathcal{O}_{Y, \underline{y}}$) corresponds to (X, D'_X) (resp. (Y, D_Y)).

Proof. Taking $\mu : (Z, D_Z) \rightarrow (X, D_X)$, one can find $D'_Z \subset D_Z$ which is no vertical component with respect to $g' : (Z, D'_Z) \rightarrow (Y, D_Y)$. Hence one chooses D'_X corresponding to D'_Z . \square

Definition 5.4. Let $f : X \rightarrow Y$ be a dominant h -strictly rational map. A dominant injective homomorphism $\mathcal{O}_{Y, \underline{y}} \rightarrow \mathcal{O}_{X, \underline{x}}$ of semi-local rings of height 1 should correspond to a dominant h -strictly rational map.

Hence by definition

$$\text{Hom}_K^{\text{dom } h\text{-st}}((X \setminus D_X), (Y \setminus D_Y)) \subset \text{Hom}_{\Gamma_K}^{\text{dom}}((\text{loc}(X, D_X)), (\text{loc}(Y, D_Y)))$$

Theorem 5.1. (Mochizuki) Let p be a prime number and K a subfield of a finitely generated field extension of \mathbb{Q}_p . Let X_K be a smooth pro-variety over K and Y_K a hyperbolic pro-curve over K . Let $\text{Hom}_K^{\text{dom}}(X_K, Y_K)$ be the set of dominant K -morphisms from X_K onto Y_K . Let $\text{Hom}_{\Gamma_K}(\Pi_{X_K}, \Pi_{Y_K})$ be the set of open, continuous group homomorphisms from Π_{X_K} to Π_{Y_K} over Γ_K up to composition with an inner automorphism arising from Δ_Y . Then the natural map

$$\text{Hom}_K^{\text{dom}}(X_K, Y_K) \rightarrow \text{Hom}_{\Gamma_K}^{\text{open}}(\Pi_{X_K}, \Pi_{Y_K})$$

is bijective.

Theorem 5.2. The natural map

$$\text{Hom}_K^{\text{dom}}(\text{loc}(X, D_X), \text{loc}(Y, D_Y)) \cong \text{Hom}_{\Gamma_K}^{\text{open}}(\Pi(\text{loc}(X, D_X)), \Pi(\text{loc}(Y, D_Y)))$$

is bijective,

Proof. Profinite version of Mochizuki's theorem above is also valid. By induction one reduces to Mochizuki's theorem. One can replace K by a function field L of dimension $\dim_K X - 1$ such that $\text{loc}(Y, D_Y)$ is defined over L . \square

6 Direct image of multiple relative log dualizing sheaves

Conjecture 6.1. *Let $f : (X, D_X) \rightarrow (S, D_S)$ be a surjective connected morphism between log varieties with $\dim X/S = n$. Assume that $D_S = 0$, $\text{var}((X, D_X)/S) = \dim S$. Then there exists a number m such that $\kappa(\det f_*(\Omega_{X/S}(\log D_X)^n)^{\otimes m}) = \dim S$.*

Theorem 6.1 (cf. Kawamata-Viehweg). *Let $f : (X, D_X) \rightarrow (S, D_Y)$ be a connected morphism between log varieties and $\dim X/S = n$. Assume that*

$$(a) D_Y = 0$$

$$(b) f_*\Omega_{X/S}^n(\log D_X) \neq 0$$

$$(c) \text{ one has } \ker(\Phi_s : T_{S,s} \rightarrow \text{Hom}(f_*\Omega_{X/S}^n(\log D_X), R^1 f_*\Omega_{X/S}^{n-1}(\log D_X))_s) = 0 \text{ for a general point } s \in S \text{ such that } f \text{ is smooth,}$$

Then $\det f_\Omega_{X/S}^n(\log D_X)$ is big. Furthermore $f_*(\Omega_{X/S}^n(\log D_X))^{\otimes m}$ is big for $m \geq 2$ if it is not zero.*

Lemma 6.1. *Let $(X, D_X) \rightarrow (S, D_S)$ be a connected surjective morphism between log varieties and $\dim X/S = n$. Assume that*

$$(a) \dim \ker(\Phi_s : T_{S,s} \rightarrow \text{Hom}(f_*\Omega_{X/S}^n(\log D_X), R^1 f_*\Omega_{X/S}^{n-1}(\log D_X))_s) > 0 \text{ for a general point } s \in S.$$

Then one has $\text{var}(X/S) \leq \dim S - \dim \ker \Phi_s$.

Proof. Let U be the open subvariety of S such that $f : X \rightarrow S$ is smooth. The variation of Hodge structure defined by $R^n f_* \mathbb{C}$ gives the Griffiths map $\Psi : U \rightarrow D/\Gamma$. The kernel of the differential $d\Psi$ is $\ker \Phi$. Hence $\text{var}(X/S) \leq \dim S - \dim \ker \Phi_s$. This will be proved by the next lemma. \square

The next lemma is a key role for proving the conjecture. One will see the complete proof using the π_1 groups due to Mochizuki's theorem later.

Lemma 6.2. *Let $X \rightarrow S$ be a connected surjective morphism between log varieties and $\dim X/S = n$. Assume that*

$$(a) f_*\Omega_{X/S}^n(\log D_X) \neq 0, \text{ hence Kodaira dimension of a general fibre of } (X, D_X)/S \text{ is non negative,}$$

$$(b) \dim \ker(\Phi_s : T_{S,s} \rightarrow \text{Hom}(f_*\Omega_{X/S}^n(\log D_X), R^1 f_*\Omega_{X/S}^{n-1}(\log D_X))) = \dim S \text{ for a general point } s \in S.$$

Then one has $\text{var}((X, D_X)/S) = 0$.

Proof. Note that $\Omega_{X/S,s}^{n-1}(\log D_X) \cong \Omega_{X/S,s}^{n-1}(\log D_X) \otimes \Omega_{X/S,s}^1(\log D_X)$ for a general $s \in S$. By assumption infinitesimal Hodge bundles are invariable, so is $f_*\Omega_{X/S}^n(\log D_X)$. Hence Kodaira-Spencer type map $\rho_s : T_{S,s} \rightarrow R^1 \Theta_{X/S}(K_{X/S} + D_X)$ is a zero map. Hence one has a splitting of the following exact sequence; $0 \rightarrow \text{Der}_{R(S)}(\mathcal{O}_{X,\underline{x}}) \rightarrow \text{Der}(\mathcal{O}_{X,\underline{x}}) \rightarrow \text{Der}(R(S)) \rightarrow 0$. One obtains the lemma using the argument of the representation of the absolute Galois group of the function field of a variety of Kodaira dimension ≥ 0 . \square

Hence one obtains the statement to be shown in the following section.

Lemma 6.3. *Let $f : (X, D_X) \rightarrow (S, D_S)$ be a dominant strictly rational map between log varieties with a general fibre of Kodaira dimension ≥ 0 with $D_S = 0$. Assume the canonical map of rings of the derivatives of the semi-local rings $\text{Der}(\mathcal{O}_{X,\underline{x}}) \rightarrow \text{Der}(R(S))$ splits. Then $\text{var}((X, D_X)/S) = 0$.*

7 Log Derivations

Theorem 7.1. (Itaka) *Let (X, D_X) be a log variety of Kodaira dimension ≥ 0 . Then $S\text{Bir}((X, D_X))$ is a group scheme locally of finite type with countable components and its neutral component is a quasi-abelian variety.*

Remark 7.1. *Assume that $\text{Out}(H)$ is a group scheme locally of finite type with countable components. Assume that the neutral component is a quasi-abelian variety A . Given $\text{Out}(H)$ the topology induced by the congruence subgroup topology on $\text{Aut}(H)$, the closure of A is a commutative group. Since A is a normal subgroup of $\text{Out}(H)$, the closure \bar{A} of A is also a normal closed subgroup of $\text{Out}(H)$. Hence $\text{Out}(H)/\bar{A}$ is a countable group, which is totally disconnected.*

Let $\phi : G \rightarrow K$ be an open continuous epimorphism of profinite groups and denote the kernel ϕ by H . $Out(H)$ is endowed with the quotient topology induced by the congruence subgroup topology. Then $\rho : K \rightarrow Out(H)$ is a continuous representation.

Assume that $Out(H)$ is a group scheme locally of finite type with countable components with respect to Zariski topology and that the neutral component with respect to Zariski topology is a quasi-abelian variety A . Forget Zariski Topology and take the closure \overline{A} of A with respect to the quotient topology on $Out(H)$ induced by the congruence subgroup topology $Aut(H)$. Then $\rho^{-1}(\overline{A})$ is a closed normal subgroup.

Lemma 7.1. $K/\rho^{-1}(\overline{A})$ is a profinite group; hence a finite group.

Proof. Since $\rho^{-1}(\overline{A}) \triangleleft_c K$, $K/\rho^{-1}(\overline{A})$ is a profinite group. $K/\rho^{-1}(\overline{A}) \rightarrow Out(H)/\overline{A}$ is injective. $Out(H)/\overline{A}$ is countable. Since there is no countable profinite group, $K/\rho^{-1}(\overline{A})$ is a finite group. \square

Pulling back an epimorphism $G \rightarrow K$ of profinite groups by a monomorphism $\rho^{-1}(\overline{A}) \rightarrow K$, one denotes by K' $\rho^{-1}(\overline{A})$ and one obtains a continuous representation $\rho : K' \rightarrow Out(H)$ and an exact sequence

$$1 \rightarrow H \rightarrow G \times_K K' \rightarrow 1.$$

The representation $\rho : K' \rightarrow Out(H)$ factors through the normal closed subgroup \overline{A} . Furthermore, the restriction of the representation $\rho : K' \rightarrow Out(H)$ to the closure of the commutator subgroup $D(K') = [K', K']$ is trivial. Hence one obtains the following lemma by the similar argument in the section 4.

Lemma 7.2. Let $f : (X, D_X) \rightarrow (S, D_S)$ be a dominant strictly rational map between log varieties with a general fibre of Kodaira dimension ≥ 0 with $D_S = 0$. Assume the canonical map of rings of the derivatives of the semi-local rings $Der(\mathcal{O}_{X, \underline{x}}) \rightarrow Der(R(S))$ splits. Then $var((X, D_X)/S) = 0$.

Thus one has the following.

Theorem 7.2. Let $f : (X, D_X) \rightarrow (S, D_S)$ be a surjective connected morphism between log varieties with $\dim X/S = n$. Assume that $D_S = 0$, $var((X, D_X)/S) = \dim S$. Then there exists a number m such that $\kappa(\det f_*(\Omega_{X/S}(\log D_X)^n)^{\otimes m}) = \dim S$.

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