An Elementary Proof of the Generalized Laplace Expansion Formula

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In this article, we generalize Laplace expansion formula to arbitrary partitions of the row and column of a square matrix. Also, we prove a determinantal formula including the minor determinants of rectangular matrices which generalize the multiplicative law of the determinant. Finally, the adjoint matrix law is generalized in this setting. The essential part lies in the definition of the sign function on the (marked) set of bijections between two finite sets. Using this 'generalized' signature, the proof is straightforward, without relying on induction. Though the subject is rather elementary, both the method and the results seem to be new.

1 Preparations

Let S_n be the symmetric group of degree n.

Definition 1.1. *Let A be a square matrix of degree n. The determinant of A is defined by*

$$|A| = \sum_{(j_1, j_2, \cdots, j_n)} \operatorname{sgn} \begin{pmatrix} 1 & 2 & \cdots & n \\ j_1 & j_2 & \cdots & j_n \end{pmatrix} a_{1,j_1} a_{2,j_2} \cdots a_{n,j_n},$$

where the summation is taken over all permutations of the column indices 1, 2, ... , n, *and* sgn *is the signature homomorphism from the symmetric group S_n to $\{\pm 1\}$.*

Now, let $I = (i_1, i_2, \cdots, i_n)$ and $J = (j_1, j_2, \cdots, j_n)$ be two permutations of n numbers which are not necessarily 1, 2, ... , n. In the following, we put $I^\sigma = (i_{\sigma(1)}, i_{\sigma(2)}, \cdots, i_{\sigma(n)})$, where $\sigma \in S_n$, and $a_{I,J} = a_{i_1,j_1}\, a_{i_2,j_2} \cdots a_{i_n,j_n}$. Then, one can define a signature function on the set of bijections between two sets with the same number of elements as follows:

Definition 1.2. *Let I and J be permutations of distinct n numbers which are not necessarily* 1, 2, ... , n. *Let σ and τ be elements of S_n. Then the sign of the map $\begin{pmatrix} I^\sigma \\ J^\tau \end{pmatrix}$ is defined by*

$$\operatorname{sgn} \begin{pmatrix} I^\sigma \\ J^\tau \end{pmatrix} = \operatorname{sgn}\sigma \, \operatorname{sgn}\tau \, \operatorname{sgn} \begin{pmatrix} I \\ J \end{pmatrix}.$$

Note that $\operatorname{sgn} \begin{pmatrix} I^\sigma \\ J^\tau \end{pmatrix}$ is defined *modulo* the value of $\operatorname{sgn} \begin{pmatrix} I \\ J \end{pmatrix}$, which, in case the two permutations I and J coincide, is naturally defined to be the unity, but otherwise there is no canonical way of definition [1].

Consider the n-term expansion

$$(x_1 + x_2 + \cdots + x_n)^n = \sum \frac{n!}{a_1!\, a_2! \cdots a_n!} x_1^{a_1}\, x_2^{a_2} \cdots x_n^{a_n},$$

where the sum is taken over all sequences $(a_1, a_2, \cdots, a_n)$ of non-negative integers with $a_1 + a_2 + \cdots + a_n = n$. There are ${}_nH_n = {}_{2n-1}C_n = \frac{(2n-1)!}{(n-1)!\, n!}$ distinct terms on the right-hand side. Furthermore, one has

$$(x_1 + x_2 + \cdots + x_n)^n \equiv n!\, x_1\, x_2 \cdots x_n.$$

modulo the ideal $(x_1^2, x_2^2, \cdots, x_n^2)$ in the ring $\mathbf{Z}[x_1, x_2, \cdots, x_n]$.

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Proposition 1.1. *Let $I = (1, 2, \cdots, n)$ and σ be a fixed element in S_n. Then*

$$|A| = \sum_{\rho \in S_n} \mathrm{sgn}\rho\, a_{I,I^\rho} = \sum_{\tau \in S_n} \mathrm{sgn}\begin{pmatrix} I^\sigma \\ I^\tau \end{pmatrix} a_{I^\sigma, I^\tau}.$$

Proof. Since $a_{\sigma(1),\tau(1)} \cdots a_{\sigma(n),\tau(n)} = a_{1,\tau\sigma^{-1}(1)} \cdots a_{n,\tau\sigma^{-1}(n)}$, we have

$$\sum_{\tau \in S_n} \mathrm{sgn}\begin{pmatrix} I^\sigma \\ I^\tau \end{pmatrix} a_{I^\sigma, I^\tau} = \sum_{\tau \in S_n} \mathrm{sgn}\tau\, \mathrm{sgn}\sigma^{-1}\, a_{I,\, I^{\tau\sigma^{-1}}} = \sum_{\rho \in S_n} \mathrm{sgn}\rho\, a_{I,\, I^\rho} = |A|,$$

where we put $\rho = \tau\,\sigma^{-1}$. □

Now, let $A_{I,J}$ designate the minor determinant of a (not necessarily square) matrix A determined by taking the n rows and n columns corresponding to the number indices in I and J, respectively, both in arbitrary order, i.e., $A_{I,J} = \sum_{\rho \in S_n} \mathrm{sgn}\,\rho\, a_{I,J^\rho}$.

Proposition 1.2. *Assigning* $\mathrm{sgn}\begin{pmatrix} I \\ J \end{pmatrix} = 1$,

$$A_{I,J} = \sum_{\rho \in S_n} \mathrm{sgn}\begin{pmatrix} I \\ J^\rho \end{pmatrix} a_{I,J^\rho} = \sum_{\tau \in S_n} \mathrm{sgn}\begin{pmatrix} I^\sigma \\ J^\tau \end{pmatrix} a_{I^\sigma, J^\tau},$$

where $\sigma \in S_n$ is a fixed permutation.

Proposition 1.3. *If $\{I_1, I_2, \cdots, I_s\}$ and $\{J_1, J_2, \cdots, J_s\}$ are two sets of element-wise distinct permutations, and if $|I_k| = |J_k| = n_k$ and $\sigma_k, \tau_k \in S_{n_k}$ for $k = 1, 2, \ldots, s$, then*

$$\mathrm{sgn}\begin{pmatrix} I_1^{\sigma_1} & I_2^{\sigma_2} & \cdots & I_s^{\sigma_s} \\ J_1^{\tau_1} & J_2^{\tau_2} & \cdots & J_s^{\tau_s} \end{pmatrix} = \mathrm{sgn}\begin{pmatrix} I_1 & I_2 & \cdots & I_s \\ J_1 & J_2 & \cdots & J_s \end{pmatrix} \prod_{k=1}^{s} \mathrm{sgn}\,\sigma_k\, \mathrm{sgn}\,\tau_k.$$

2 Results

It is well-known that a determinant allows Laplace expansion formula. Note that, in our notation, it states that

$$|A| = \sum_{(J_1|J_2)} \mathrm{sgn}\begin{pmatrix} I_1 & I_2 \\ J_1 & J_2 \end{pmatrix} A_{I_1,J_1} A_{I_2,J_2},$$

where $(I_1|I_2)$ is a fixed partition of $\{1, 2, \cdots, n\}$ of size (n_1, n_2), and $(J_1|J_2)$ runs through all the partitions of $\{1, 2, \cdots, n\}$ of size (n_1, n_2). Essencially, it comes from the equality $n! = {}_nC_p \times p! \times q!$ for $n = p + q$.

Similarly, from the formula for the multinomial coefficient $n! = {}_nC_{n_1, n_2, \cdots, n_s} \times n_1! \times n_2! \times \cdots \times n_s!$, where $n = n_1 + n_2 + \cdots + n_s$, one can derive the following *new* expansion formula for the determinant.

Theorem 2.1. *Let $(n_1, n_2, \cdots, n_s)$ be any sequence of positive integers with $n_1 + n_2 + \cdots + n_s = n$, and A a square matrix of degree n. Then one obtains the following formula:*

$$|A| = \sum_{(J_1|J_2|\cdots|J_s)} \mathrm{sgn}\begin{pmatrix} I_1 & I_2 & \cdots & I_s \\ J_1 & J_2 & \cdots & J_s \end{pmatrix} A_{I_1,J_1} A_{I_2,J_2} \cdots A_{I_s,J_s},$$

where $(I_1|I_2|\cdots|I_s)$ is a fixed partition of $\{1, 2, \cdots, n\}$ of size $(n_1, n_2, \cdots, n_s)$, and $(J_1|J_2|\cdots|J_s)$ runs through all the partitions of $\{1, 2, \cdots, n\}$ of size $(n_1, n_2, \cdots, n_s)$ [2].

Proof. From Definition 1.1 one has

$$\begin{aligned} |A| &= \sum_{(J_1|\cdots|J_s)} \sum_{\sigma_1 \in S_{n_1}} \cdots \sum_{\sigma_n \in S_{n_s}} \mathrm{sgn}\begin{pmatrix} I_1 & \cdots & I_s \\ J_1^{\sigma_1} & \cdots & J_s^{\sigma_s} \end{pmatrix} a_{I_1,J_1^{\sigma_1}} \cdots a_{I_s,J_s^{\sigma_s}} \\ &= \sum_{(J_1|\cdots|J_s)} \mathrm{sgn}\begin{pmatrix} I_1 & \cdots & I_s \\ J_1 & \cdots & J_s \end{pmatrix} \sum_{\sigma_1 \in S_{n_1}} \mathrm{sgn}\,\sigma_1 a_{I_1,J_1^{\sigma_1}} \cdots \sum_{\sigma_s \in S_{n_s}} \mathrm{sgn}\,\sigma_s a_{I_s,J_s^{\sigma_s}}. \end{aligned}$$

Here we have used Proposition 1.3. □

For $\alpha \geq n$, consider an α-term development

$$(x_1 + x_2 + \cdots + x_\alpha)^n = \sum \frac{n!}{n_1!n_2!\cdots n_\alpha!}\, x_1^{n_1} x_2^{n_2} \cdots x_\alpha^{n_\alpha},$$

where the sum is taken over all the sequences $(n_1,\, n_2,\, \cdots,\, n_\alpha)$ of non-negative integers with $n_1+n_2+\cdots+n_\alpha = n$. There are ${}_\alpha H_n$ distinct terms on the right-hand side. One has, furthermore, up to an ideal $(x_1^2, x_2^2, \cdots, x_\alpha^2)$ in $\mathbf{Z}[x_1,\, x_2,\, \cdots,\, x_\alpha]$,

$$(x_1 + x_2 + \cdots + x_\alpha)^n \equiv \sum_{1\leq i_1<i_2<\cdots<i_n\leq\alpha} n!\, x_{i_1}\, x_{i_2} \,\cdots\, x_{i_n},$$

where there are ${}_\alpha C_n$ distinct terms on the right-hand side.

Theorem 2.2. *Theorem 2.2. Let $A,\, B,\, \cdots,\, D$ be matrices of types (n,α), (α,β), $\cdots$, (γ,n), respectively. Then one has*

$$|A\,B\,\cdots\,D| = \sum_J \sum_K \cdots \sum_L A_{I,J}\, B_{J,K}\, \cdots\, D_{L,I},$$

where $I = (1,\, 2,\, \cdots,\, n)$ and $J,\, K,\, \cdots,\, L$ run through all the combinations of size n of $\{1,\, 2,\, \cdots,\, \alpha\}$, $\{1,\, 2,\, \cdots,\, \beta\}$, $\cdots$, $\{1,\, 2,\, \cdots,\, \gamma\}$, respectively[3].

Proof. From Definition 1.1, one has

$$\begin{aligned} |A\,B\cdots D| &= \sum_{\mu\in S_n} \mathrm{sgn}\mu \left(\sum_J \sum_K \cdots \sum_L \sum_{\sigma\in S_n} \sum_{\tau\in S_n} \cdots \sum_{\lambda\in S_n} a_{I,J^\sigma} b_{J^\sigma,K^\tau} \cdots d_{L^\lambda,I^\mu} \right) \\ &= \sum_J \sum_K \cdots \sum_L \left(\sum_{\sigma\in S_n} \sum_{\tau\in S_n} \cdots \sum_{\lambda\in S_n} \sum_{\mu\in S_n} \mathrm{sgn} \begin{pmatrix} I \\ I^\mu \end{pmatrix} a_{I,J^\sigma} b_{J^\sigma,K^\tau} \cdots d_{L^\lambda,I^\mu} \right). \end{aligned}$$

Since $\begin{pmatrix} I \\ J^\sigma \end{pmatrix} \circ \begin{pmatrix} J^\sigma \\ K^\tau \end{pmatrix} \circ \cdots \circ \begin{pmatrix} L^\lambda \\ I^\mu \end{pmatrix} = \begin{pmatrix} I \\ I^\mu \end{pmatrix}$, we have

$$\begin{aligned} &\sum_{\sigma\in S_n} \sum_{\tau\in S_n} \cdots \sum_{\lambda\in S_n} \sum_{\mu\in S_n} \mathrm{sgn} \begin{pmatrix} I \\ I^\mu \end{pmatrix} a_{I,J^\sigma} b_{J^\sigma,K^\sigma} \cdots d_{L^\lambda,I^\mu} \\ &= \sum_{\sigma\in S_n} \sum_{\tau\in S_n} \cdots \sum_{\lambda\in S_n} \sum_{\mu\in S_n} \mathrm{sgn} \begin{pmatrix} I \\ J^\sigma \end{pmatrix} a_{I,J^\sigma}\, \mathrm{sgn} \begin{pmatrix} J^\sigma \\ K^\tau \end{pmatrix} b_{J^\sigma,K^\tau} \cdots \mathrm{sgn} \begin{pmatrix} L^\lambda \\ I^\mu \end{pmatrix} d_{L^\lambda,I^\mu} \\ &= \sum_{\sigma\in S_n} \mathrm{sgn} \begin{pmatrix} I \\ J^\sigma \end{pmatrix} a_{I,J^\sigma} \sum_{\tau\in S_n} \mathrm{sgn} \begin{pmatrix} J^\sigma \\ K^\tau \end{pmatrix} b_{J^\sigma,K^\tau} \cdots \sum_{\mu\in S_n} \mathrm{sgn} \begin{pmatrix} L^\lambda \\ I^\mu \end{pmatrix} d_{L^\lambda,I^\mu} \\ &= A_{I,J}\, B_{J,K}\, \cdots\, D_{L,I}, \end{aligned}$$

hence one obtains the consequence. □

Definition 2.1. *Let A be a square matrix of degree n, and $A_{I,J}$ the minor determinants of degree r. The square matrix (A_{IJ}) of degree ${}_nC_r$ is said to be the 'derived matrix' of A of rank r and denoted by* $\mathrm{der}A$.

Theorem 2.3. *Let A be a square matrix of degree n, and let $\mathrm{der}\,A = (A_{I_1,J_1})$ be the derived matrix of A of rank r. Then*

(1) $\quad {}^t\mathrm{adj}(\mathrm{der}\,A) = \left(\mathrm{sgn} \begin{pmatrix} I_1 & I_2 \\ J_1 & J_2 \end{pmatrix} A_{I_2,J_2} \right)$, *and*

(2) $\quad \mathrm{der}\,A\,\mathrm{adj}(\mathrm{der}\,A) = \mathrm{adj}(\mathrm{der}\,A)\,\mathrm{der}\,A = |A|\, 1_{{}_nC_r}$,

where $\mathrm{adj}(\cdot)$ *means the adjoint matrix.*

The proof is left out.

Comments

[1] In what follows, it is convienient to define $\mathrm{sgn} \begin{pmatrix} I \\ J \end{pmatrix} = 0$ if either I or J contains repeated elements.

[2] There are ${}_nC_{n_1,\, n_2,\, \cdots,\, n_s} = \dfrac{n!}{n_1! \times n_2! \times \cdots \times n_s!}$ distinct terms on the right-hand side in this expansion.

[3] There are ${}_\alpha C_n \times {}_\beta C_n \times \cdots \times {}_\gamma C_n$ distinct terms on the right-hand side in this expansion.