

Kronecker's Intuitionism and Fujisawa's Enumerationism —A Historical Consideration—

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We consider how Rikitaro Fujisawa's pedagogic thoughts of mathematics were affected by Leopold Kronecker's lecture in 1887. The focus is on the comparison between the so-called 'enumerationism' and the construction of natural numbers carried out in Kronecker's lecture "On the Concept of Numbers", the former being said to have determined the order of materials in the first national arithmetic textbook compiled by the Japanese government.

1 R.Fujisawa and L.Kronecker

After the Restoration of Imperial Power in 1868, the Meiji era government put up modern school system. In 1877, the University of Tokyo was settled up. The department of mathematics therein started to study European-style mathematics with the two main staff, Professors Dairoku Kikuchi (1855-1917) and Rikitaro Fujisawa (1861-1933) [2].

Fujisawa studied mathematics, physics and astronomy, and graduated from the University of Tokyo in 1882. From 1883 to 1884, he studied in Berlin University under Leopold Kronecker. After coming back to Japan, in 1899, he gave a series of lectures about the pedagogy of elementary mathematics at a summer workshop. On the third day, Fujisawa mentioned the so-called enumerationism, starting with the prehistory of his first encounter with that discipline [1].

...and in the last place occurred enumerationism. It was about 15 years ago from now when Tank and Kniling started to advocate it. On the other hand, I myself did not know much about their movement, but I just wanted to get a head start on Europeans and Americans by this enumerationism. The reason was that although it was already known for long in higher mathematics that numbers originated from counting, no one ever applied it to elementary school education. When I was about to leave Berlin, I was given a piece of thesis entitled "On the Concept of Numbers" as a farewell gift from my teacher, a man named Kronecker. It was a paper on the concept of numbers, and it was written so beautifully that since then I have been hoping to carry out this theory in elementary education. That idea itself was right because it was already believed right firmly by specialists, but at that time the social circumstances were not prepared enough to give me a chance to carry it out. But I personally believed that it must be a feasible theory...

We can understand from these reminiscences that the idea that "numbers originate from (the act of) counting" was already well-known as one of the methods to found the concept of numbers in Europe at that time. Fujisawa says that he, independently of the European scholars, had reached the idea of applying it to the field of education and that he eventually carried it out.

Fujisawa then goes on to explaining the essential points in the concrete methods of teaching number concept, but before looking at them, let us take a look at how Kronecker's paper had been organized, which, according to Fujisawa, left a great impression in his mind on the beauty of enumerationism.

2 Kronecker's Lecture on the Concept of Numbers

Leopold Kronecker (1823.12.7-1891.12.29) was a professor at Berlin University since 1883. His mathematical work ranged from arithmetic, algebra through analysis. His contribution to the theory of elliptic functions and ζ functions were prominent. As for the foundations of mathematics, he denied the bold arguments of set theory.

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Received Sept. 12, 2002

He insisted that the whole body of mathematics should be constructed rigidly on the basis of intuition of natural numbers.

His standpoint is historically named 'intuitionism'. In intuitionism, both truths and objects of mathematics do not stand independent of mind, but they can be grabbed directly by mental activity. Later, Brouwer defined intuitionism more strictly. Brouwer did not allow the use of the law of the excluded middle. The assertions of Brouwer showed strict conflict with those of Dedekind and Cantor. It is well-known that Kronecker once said, "Natural numbers were created by God. All others were the creation of men", showing strict contrast to the 'freedom' assertion made by Cantor and J.W.R.Dedekind [7].

Looking into Kronecker's collected works [3], one finds that his paper "Ueber den Zahlenbegriff", published in the Crelle Journal in 1887, stands out. Unlike other technical papers of mathematics, this paper was based on his article submitted to a bulletin of philosophical lectures given at the fiftieth anniversary of Eduard Zeller's laureation.

The paper can be divided into three parts:

- (1) Preface
- (2) from §1 to the first part of §5
- (3) latter part of §5

He starts Preface with the relation between philosophy and mathematics. He states that philosophy, as its role, has to define the basic concepts for sciences to start with, and it has the same role for mathematics, too, and he puts up numbers, space, and time as the basic concepts of mathematics [8]. Then, he declares that he will treat the foundations of the number concept and elicitation of basic properties of numbers.

He also cites the famous words of Gauss, "Mathematics is the queen of sciences, and arithmetic is the queen of mathematics", and mentions the role of arithmetic in mathematics, expressing his far-reaching wishes to someday found the whole body of mathematics rigidly on the basis of arithmetic.

... and, I also believe that, someday in the future, we will be successful in 'arithmetizing' the whole body of these mathematical sciences, i.e. to found it entirely on the concept of numbers interpreted in the most narrow meaning of it.

Sections from §1 to the first part of §5 are devoted to the construction of the concept of numbers, on which Enumerationism puts its basis. At first, he mentions that

I find the natural starting point for developing the concept of numbers in **ordinal numbers**.

Here, ordinal numbers are just

some kind of symbols put in a line in a certain, fixed order,

and

we can associate these symbols to a class of any objects which are different and distinguishable from each other.

This action of association is what is called 'to count'. As a result of counting, one cardinal number is determined corresponding to the ordinal number used at last. This is the 'number' expressing the whole body of objects to be counted.

Thus, in Kronecker's construction of numbers, there is at first a set of something called ordinal numbers, which reside in mind, and the concept of cardinal numbers, which express quantities, will emerge through the (intuitively very natural) action of counting the objects using these ordinals. The definition of addition and multiplication, together with the law of commutativity, will be naturally explained through this action of counting. It is explained in §2 that the whole number of counted objects does not depend on the order of counting. Then, addition and its properties are explained in §3, and multiplication and its properties are explained in §4.

At the beginning of §5, the concept of indeterminate is introduced. An indeterminate x is defined to be nothing but a letter, any arithmetic including which can be carried out similarly as ordinary arithmetic. But, then, all of a sudden, the extension of the number concept from natural numbers to integers and fractions will be made. Namely, the concept of negative integers is defined by the transition to the congruent classes modulo $(x + 1)$ in the sense of Gauss, while the concept of fractions or rational numbers is introduced by considering the congruent classes modulo $(mx_m - 1)$ for each integer m .

Thus the extension of numbers is performed here very abstractly and formally, and there is apparently no connection with the concept of quantity in its perspective. This way of transition bears close resemblance with the definition of an algebraic number using its minimal polynomial.

Although Fujisawa says that he was much inspired by this paper of Kronecker's in inventing his mathematics pedagogy of enumerationism, negative integers and fractions are not introduced via congruent classes in the textbook of middle school algebra edited by him. But it seems that under the surface of his textbook, there was an idea that this kind of rigidity should be desirable in the discipline of algebra.

The latter part of §5 is a very long discussion on the existence of real roots of an algebraic equation, providing a foundation for the theory of algebraic numbers, which was the author's central concern. We do not go into this part, because it is too technical for the purpose of our present concern.

3 Fujisawa Explains Enumerationism for School Teachers

Now, we would go back to the lecture hall in the summer of 1899, where Fujisawa explains the essential points of his enumerationism before the audience of nearly 300 teachers from normal schools and middle schools amongst Japan.

Now, how do we carry out enumerationism? At the very beginning stage, we must train our students to count properly. And the method of that training should, of course, be based on enumerationism.

Fujisawa explains first that one plus one is two, two plus one is three, three plus one is four, \dots , and thus adding continually, we get natural numbers, and that this is also possible by folding fingers one by one.

\dots so this ability of counting is the nature of human beings \dots and it seems that mathematicians, especially as middle school teachers, need not go further into the detail than this depth.

Then Fujisawa states, "Thus adding one repeatedly is called **counting**, and what we get from counting is called the **numbers**, and the numbers are the basis of mathematics".

This explanation seems to be almost parallel to Kronecker's paper, although the distinction between ordinal and cardinal numbers is not emphasized here at all.

There was a strict distinction between the numbers as symbols used to count and the objects to be counted in Kronecker's paper, but in Fujisawa's lecture, this distinction is also fogged away. In Kronecker's lecture, natural numbers are arbitrary but fixed set of linearly ordered symbols, and the act of counting is nothing else but making a 1:1 correspondence between the set of integers to some length and the set of objects. In Fujisawa's explanation, natural numbers are what we get from counting continuously, and to count is 'to add one repeatedly'. In Kronecker's paper, addition is defined by renaming the objects, or to take a certain set of numbers as the object of counting. In Fujisawa's explanation, addition of one comes at first, and counting is explained by repeated application of this action.

Fujisawa then explains

With natural numbers in hand, one can theoretically count any number, however large it might be. But, that being ambiguous, it is natural to wonder if there might be other ways of doing it more practically. It is from this consideration that the idea of arithmetic or the system of decimal enumeration comes out.

He then says that after getting natural numbers, the next thing pupils should practice is to practice to count by adding a certain fixed number, such as to count by adding twos like

$$1, 3, 5, 7, 9, 11, 13, 15, \dots$$

or to count by adding threes like

$$1, 4, 7, 10, 13, 16, 19, 22, \dots$$

or

$$2, 5, 8, 11, 14, 17, 20, 23, \dots$$

and so forth. He states that in enumerationism, it is important (to increase the easiness of counting) to divide into groups, and dividing into groups of tens is the basis of decimal system. He also points out that neat and clean things are good to see, so it is important to habituate pupils to write formulas neatly.

4 What was Fujisawa's Enumerationism?

In the last section we pointed out some differences between Kronecker's foundations of natural numbers and that of Fujisawa's. To sum up in a word, Kronecker's paper was theoretical, while Fujisawa's lecture was casual. It seems that Fujisawa tried to make his lecture practical to the audience at the risk of losing rigidity.

In the first national textbook of elementary arithmetic, which was edited by Fujisawa, teachers were encouraged to bring in various things to count. What to count is important in understanding the meaning of numbers, and how to count is more technical a question. But it is no wonder teachers in the classroom paid less attention to the meaning and objective of counting, and they get interested in just training pupils to count accurately, effectively, fast and skillfully. It should be pointed out here that enumerationism had, at the very beginning, the risk of becoming just practice of vocalizing numbers like singing in chorus.

To conclude our consideration, we would like to point out that, inspired by Kronecker's paper, Fujisawa constructed the process of number acquisition and progress of skills in arithmetic on the basis of adding by one. In his pedagogical course of arithmetic, 'addition by one' was the primitive process, the building block of the system. There was an affect of reductionism.

This process being easy, Fujisawa's plan attained some progress. Indeed, small children like to count vocally, before they understand the meaning of numbers.

5 A Historical Consideration

In this paper, we looked comparatively at Kronecker's intuitionism and Fujisawa's enumerationism. On one hand, Kronecker attempted to construct the concept of numbers on the basis of the natural intuition of natural numbers. He aimed eventually at the 'arithmetization' of the whole body of mathematics. On the other hand, Fujisawa proposed enumerationism as the fundamental principle in creating the curriculum-textbook in the elementary and middle education. As a historical momentum, the former movement had strong influence on the latter.

In the history of European mathematics, Kronecker's naive form of intuitionism was later absorbed in Hilbert's standpoint of 'finite mathematics'. For Kronecker, natural numbers were the fundamental object of mathematics, and nothing has changed about it even now, but the meaning of it has changed dramatically. Mathematics has extended to include the theory of sets and logic.

It is interesting to note that in the field of educational science, the idea of enumerationism occurred almost simultaneously at different places in the world. People in the 19th century throughout the world had faced a big social change and they felt strongly that new scientific knowledge could help them substantially.

In Japan, mathematics education as a modern nation began with the series of Fujisawa's enumerationism textbooks in elementary schools. Historically speaking, it turned out to be not effective for the people in general, and the era of a new textbook called the 'green cover' follows.

Fujisawa's talk at the workshop reveals his liberal character. For example, he said, "In the future when psychology is more developed than today, there will surely be new scientific facts, for example, how the children's brain grows, to take into mathematics education." On the other hand, he was very strict on his doctrine, as he continued, "but my enumerationism will never give way." He thought that science and society should improve, but a scholar should not change his doctrine for life [6].

Consideration on how the history of mathematics was involved in the history of mathematics education is directly connected with the present problem of what the educational study of mathematics should take in from other modern sciences, including, of course, mathematics itself.

Acknowledgement This research is based on the literature search during the 2001 Kiyoshi Yokochi Library International Seminar held in Beijing, from April 29 to May 6, 2001. I would like to thank Professor Kiyoshi Yokochi for giving me an opportunity to study in the seminar with his stimulating discussions and Beijing Normal University for the hospitality during the stay. I am also indebted to Professor Kiyosi Yamaguti for his enduring encouragement.

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- [6] In this paper, we focused only on the educational doctrine of Fujisawa. He was also an outstanding mathematician in the field of differential equations. Surprisingly, he traveled a lot to schools all over Japan, and gave very concrete, enlightening and practical advice to teachers. For example, he said that teachers should not only study mathematics, but also study psychology, and he recommended writings of Herbart. It seems that not enough light is thrown yet on Fujisawa himself as a mathematician, as a teacher, and as a total person.
- [7] In Heinrich Weber's memorial message, we can find the following famous words of Kronecker.

Die ganzen Zahlen hat der liebe Gott gemacht. Alles anderes ist das Menschenwerk.
- [8] For Kronecker, mathematics consisted of three main parts, i.e., arithmetic, geometry, and physics, where arithmetic was the unified area of algebra and analysis. For him, arithmetic was a science of numbers, geometry a science of numbers and space, and physics a science of numbers, space and time.