

Fourier-Deligne-Sato Transformation

Kazuhisa MAEHARA*

In this article we experimentally reconstruct Hodge theory by means of lifting of Frobenius action and Fourier-Deligne-Sato transformation suppressing inner couplings. We based on the argument of second proofs of Weil conjecture by Laumon and proposed the substitution of Tate twist in ℓ -adic Hodge theory.

1 Introduction

Hodge theory is classically founded by Hodge-Kodaira-de Rham in making use of harmonic integral theory([6]). This method had dominated algebro-analytic geometry long period. In 1974 Deligne proved Weil conjecture in the field of Grothendieck([9]). One of its many applications is the strong Lefschetz formula, which implies Hodge decomposition. Hodge decomposition theorem plays a central role in Hodge theory. A pure algebraic approach is succeeded by K.Kato using Fontaine-Messing's idea([23]). These theories depend deeply upon positive characteristic geometry. In the category of characteristic zero Faltings proved Hodge-Tate decomposition for the first time in purely algebraic way([7]). Our approach is as follows: Weil treated Hodge decomposition in the special case of a torus by using Fourier transformation in his book([4]). Griffiths and Harris wrote Hodge theory using classical Fourier transformation in their book "Principles of Algebraic Geometry"([5]). Laumon took a new route of a proof of Weil conjecture by taking Fourier-Deligne transformation instead of the method of Hadamard-de La Vallée-Poussin using classical Fourier-Laplace transformation([8], [10]). As to Fourier-Deligne transformation in the algebraic geometry of positive characteristic is Brylinski-Malgrange-Verdier's geometric Fourier transformation([3]) or Fourier-Sato transformation([13]) in the algebraic analysis founded by Sato. The Frobenius action is needed in positive characteristic case so we lift it in the characteristic zero case. Usual $\mathbb{Q}(1) = \mathbb{Q}(2\pi i)$ does not work well under geometric Fourier transformation and lifted Frobenius action. We propose its substitution later. These are all tools we need except local Fourier transformation([11]). We define the q -purity after ι -purity. We deduce the q -purity theorem on a vector line, which implies the general q -purity theorem on complex varieties and the strong Lefschetz theorem due to Deligne's work([9]).

2 Frobenius action

2.1 Lifting

We need some definitions and lemmas for lifting Frobenius action([12]).

Definition 2.1. Let A' be a ring and G a group operating on A' . For a prime ideal P' of A' the subgroup of the elements $\sigma \in G$ such that $\sigma P' = P'$ is said to be the decomposition group of P' . One denotes it by $G^Z(P')$. The invariant ring of A' by $G^Z(P')$ is said to be the decomposition ring of P' .

It is the canonical homomorphism $G^Z(P') \rightarrow \text{Aut}(A'/P')$, whose image is denoted by Γ_0 . For $\sigma \in G^Z(P')$ the endomorphism $x \rightarrow \sigma x$ of A' induces $z \rightarrow \sigma z$ of A'/P' .

Definition 2.2. The subgroup of $G^Z(P')$ which is the kernel of the canonical homomorphism is said to be the inertia group of P' and one denotes it by $G^T(P')$. The invariant ring of A' by $G^T(P')$ is said to be the inertia group of P' .

Note that $(A'/P')^{\Gamma_0} = A^Z/(P' \cap A^Z)$.

Proposition 2.1. ([12]) Let k be a field, $S = \text{Spec } k$ and Ω an algebraically closed extension of k . Let $a \in S$ be a geometric point $\text{Spec } \Omega \rightarrow S$. Let \bar{k} be the algebraic closure of k in Ω . Then there exists a canonical isomorphism $\pi_1(S, a) \cong \text{Gal}(\bar{k}/k)$ as topological groups.

* (Received Sept. 13, 1999), Tokyo Institute of Polytechnics, 1583 Iiyama Atsugi, Kanagawa 243-02, Japan, E-mail address maehara@gen.t.kougei.ac.jp

Proposition 2.2. ([15]) *Let Y be locally noetherian. Let $f : X \rightarrow Y$ be a proper morphism with a geometrically connected fibre and $y_0, y_1 \in Y$ such that y_0 is a specialization of y_1 . Let $\overline{X}_0, \overline{X}_1$ the geometric fibres of X corresponding to given algebraically closed extensions of $k(y_0), k(y_1)$ and $\overline{a}_0, \overline{a}_1$ geometric points of $\overline{X}_0, \overline{X}_1$, respectively. Then there exists a specialization homomorphism up to an inner automorphism*

$$\pi_1(\overline{X}_1, \overline{a}_1) \rightarrow \pi_1(\overline{X}_0, \overline{a}_0)$$

This is surjective if f is separable.

(cf. [14] pp.24-35) Hence one obtains the lifting of Frobenius action.

Let K be a finite extension of \mathbb{Q} , \overline{K} the algebraic closure in \mathbb{C} and Frob_q a lifting of Frob_q .

2.2 Purity of weight w

Definition 2.3. *Let X be a scheme of finite type defined over \mathbb{F}_q . A smooth $\overline{\mathbb{Q}}_\ell$ -sheaf F over X is said to be ι -pure if there exists a real number w such that for every closed point x of X and every proper value α of Frob_x acting on F , one has*

$$|\iota(\alpha)| = q^{\frac{\deg(x)w}{2}};$$

this number w is called the ι -weight of F .

A $\text{Gal}(\overline{k}/\mathbb{F}_q)$ -module H is also said to be ι -pure of ι -weight $w \in \mathbb{R}$ if for every proper value α of Frob_q which acts on H one has $|\iota(\alpha)| = q^{\frac{w}{2}}$.

Definition 2.4. *Let X be a scheme of finite type defined over K . A smooth \overline{K} -sheaf F over X is said to be q -pure if there exists a real number w such that for every closed point x of X and every proper value α of Frob_x acting on F , one has*

$$|\alpha| = q^{\frac{\deg(x)w}{2}};$$

this number w is called the weight of F . Here one denotes by $\text{Frob}_x = \text{Frob}_{q^{\deg(x)}}$.

In particular, a $\text{Gal}(\overline{K}/K)$ -module H is also said to be q -pure of weight $w \in \mathbb{R}$ if for every proper value α of Frob_q which acts on H one has $|\alpha| = q^{\frac{w}{2}}$.

3 Theorems

3.1 \mathbb{Q}_ℓ -case

We recall the main theorem of Deligne's ([9]).

Theorem 3.1 (Deligne's Purity Theorem over a curve). *Let X be a projective smooth curve over \mathbb{F}_q , $j : U \hookrightarrow X$ a dense open and F a $\overline{\mathbb{Q}}_\ell$ -sheaf which is smooth and ι -pure with ι -weight $w \in \mathbb{Z}$ over U . Then the $\text{Gal}(\overline{k}/\mathbb{F}_q)$ -module $H^i(X \otimes_{\mathbb{F}_q} \overline{k}, j_* F)$ is ι -pure with ι -weight $w + i$ for every integer i .*

Corollary 3.1 (Deligne's Purity). *Let X be a proper and smooth variety over \mathbb{F}_q . For each i , the characteristic polynomial*

$$\det(t - \text{Frob}_q, H^i(X \otimes_{\mathbb{F}_q} \overline{k}, \overline{\mathbb{Q}}_\ell))$$

is independent of $\ell \neq p$ with integral coefficient. The complex roots α of this polynomial have an absolute value $|\alpha| = q^{i/2}$.

3.2 \mathbb{C} -case

The following theorem is an analogy of Deligne's theorem.

Theorem 3.2 (Purity Theorem over a curve). *Let X be a projective smooth curve over K , $j : U \hookrightarrow X$ a dense open and F a \mathbb{C} -sheaf which is smooth and q -pure with weight $w \in \mathbb{Z}$ over U . Then the $\text{Gal}(\overline{K}/K)$ -module $H^i(X \otimes_K \overline{K}, j_* F)$ is q -pure with weight $w + i$ for every integer i .*

Corollary 3.2 (Purity Theorem). *Let X be a proper and smooth variety over K . The characteristic polynomial*

$$\det(t - \text{Frob}_q, H^i(X \otimes_K \overline{K}, \mathbb{C}))$$

is of integral coefficient. The complex roots α of this polynomial have an absolute value $|\alpha| = q^{i/2}$.

4 Proofs

We will represent a proof of this theorem after Laumon by Fourier-Deligne transformation([8]). The first tool is the deepest criterion of purity showed by Deligne.

Definition 4.1. Let X be a scheme of finite type over \mathbb{F}_q . A $\bar{\mathbb{Q}}_\ell$ -sheaf F over X is said to be ι -real if for every closed point $x \in X$, the characteristic polynomial

$$\iota \det(t - \text{Frob}_x, F) \in \mathbb{C}[t]$$

has real coefficients.

Definition 4.2. Let X be a scheme of finite type over K . A \mathbb{C} -sheaf F over X is said to be q -real if for every closed point $x \in X$, the characteristic polynomial

$$\det(t - \text{Frob}_x, F) \in \mathbb{C}[t]$$

has real coefficients.

Theorem 4.1. ([9]) The smooth irreducible sub-quotients of any ι -real $\bar{\mathbb{Q}}_\ell$ -sheaf over a smooth geometrically connected curve over \mathbb{F}_q are ι -pure. Reciprocally, every $\bar{\mathbb{Q}}_\ell$ -sheaf F , which is ι -pure with an integral ι -weight w , is a direct factor of a smooth ι -real $\bar{\mathbb{Q}}_\ell$ -sheaf F , say, $F \oplus F^\vee(-w)$.

We obtain the following statement in the same argument above.

Theorem 4.2. The smooth irreducible sub-quotients of any real \mathbb{C} -sheaf over a smooth geometrically connected curve over K are pure. Reciprocally, every \mathbb{C} -sheaf F , which is q -pure with an integral weight w , is a direct factor of a smooth real \mathbb{C} -sheaf F , say, $F \oplus F^\vee(-w)$.

Proof. This is obtained by interpretation. □

The second tool is the study of the local monodromy of ι -pur $\bar{\mathbb{Q}}_\ell$ over the curves([9], [8]).

Lemma 4.1 (Deligne). Let X be a smooth and geometrically connected curve over \mathbb{F}_q and $j : U \hookrightarrow X$ a dense open complement of a finite set S of closed points of X . If F is a smooth ι -pure $\bar{\mathbb{Q}}_\ell$ -sheaf with ι -weight $w \in \mathbb{R}$ over U , then, for any $s \in S$ and for every proper value α of Frob_s acting over $(j_* F)_s^-$ (resp. $(R^1 j_* F)_s^-$), one has the estimation

$$|\iota(\alpha)| \leq q^{\deg(s)w/2} \\ (\text{resp. } |\iota(\alpha)| \geq q^{\deg(s)(w+2)/2}).$$

We recall briefly the proof in order to translate into our case.

Proof. By Grothendieck formula

$$\prod_{x \in |X_0|} \iota \det(1 - F_x t^{\deg x}, \otimes^{2k} \mathcal{F})^{-1} = \frac{\iota \det(1 - \text{Ft}, H_c^1(X, \otimes^{2k} \mathcal{F}))}{\iota \det(1 - \text{Ft}, H_c^2(X, \otimes^{2k} \mathcal{F}))}$$

the statement to $(j_* F)_s^-$ is proved by Deligne([9]). The duality

$$(j_* F^\vee)_s^- \otimes (R^1 j_* F)_s^- \rightarrow \bar{\mathbb{Q}}_\ell(-1)$$

gives a proof of the second part. □

Hence we need the following formula.

Lemma 4.2 (Grothendieck formula). Let X_0 be a complex variety and \mathcal{F}_0 a \mathbb{C} -sheaf over X_0 . The following equality holds in complex formal series

$$\prod_{x \in |X_0|} \det(1 - \text{Frob}_x t^{\deg x}, \mathcal{F})^{-1} = \prod_i \det(1 - \text{Frob}_{q^i}, H_c^i(X, \mathcal{F}))^{(-1)^{i+1}}$$

Proof. Obvious([9]). \square

This function has the following property.

Proposition 4.1. *If for $x \in |X_0|$, the proper values α of Frob_x on \mathcal{F} satisfy that the weights of α are not more than β , the product $\prod_{x \in |X_0|} \det(1 - \text{Frob}_x t^{\deg x}, \mathcal{F})^{-1}$ converges for $|t| < q^{-\frac{\beta}{2} - \dim X_0}$, does not have neither zero nor pole in this disk.*

Proof. One can obtains the proof by the convergence of the geometric series $\sum_n q^{n(\dim X_0 + \frac{\beta}{2})} \cdot |t|^n$. \square

We remind of Laumon's consideration. The problem is as follows given a projective variety V over a field k of characteristic $p > 0$, a prime number $\ell \neq p$ and a complex of ℓ -adic sheaves K over V , one can deduce some global informations on the ℓ -adic cohomology $R\Gamma(V \otimes_k \bar{k}, K)$ from local informations, where \bar{k} is an algebraic closure of k . The essential case is $V = \mathbb{P}_k^1$. To analyze $R\Gamma(\mathbb{P}_k^1, \bar{k}, K)$ one makes use of a deformation with a parameter of this complex after the method of Witten. More presicely speaking, one fix a non trivial additive character $\psi : \mathbb{F}_p \hookrightarrow \bar{\mathbb{Q}}_\ell^\times$. For every $y \in \bar{k}$, the covering of Artin-Schreier of the affine line \mathbb{A}_k^1 by the equation

$$t^p - t = yx,$$

where x is the standard coordinate over \mathbb{A}_k^1 and the character ψ of the group of Galois \mathbb{F}_p of this covering induce a ℓ -adic-sheaf $\mathcal{L}_\psi(yx)$, smooth of rank 1 over \mathbb{A}_k^1 , whose extension by 0 to the whole \mathbb{P}_k^1 . Then the deformation on question of the complex $R\Gamma(\mathbb{P}_k^1, K \otimes \bar{\mathcal{L}}_\psi(yx))$, indexed by the $y \in \bar{k}$. In fact, $R\Gamma(\mathbb{P}_k^1, K)$ makes itself devissage in the fibre of K at the point ∞ of \mathbb{P}_k^1 and $R\Gamma_c(\mathbb{A}_k^1, K)$ and for $y = 0$ one has

$$R\Gamma(\mathbb{P}_k^1, K \otimes \bar{\mathcal{L}}_\psi(yx)) = R\Gamma_c(\mathbb{A}_k^1, K).$$

The full force of this deformation comes from the relation very close to the geometric transformation of Fourier-Deligne. Deligne defined for each non-trivial additive character $\psi : \mathbb{F}_p \hookrightarrow \bar{\mathbb{Q}}_\ell^\times$ an involution \mathcal{F}_ψ over the derived category of ℓ -adic sheaves over \mathbb{A}_k^1 , which is a geometric version of the classical transformation of Fourier over the functions $f : \mathbb{F}_p \rightarrow \bar{\mathbb{Q}}_\ell$:

$$\hat{f}(y) = \sum_{x \in \mathbb{F}_p} f(x) \psi(yx).$$

For every $y \in \bar{k}$ looked as a geometric point of \mathbb{A}_k^1 , the complex $R\Gamma(\mathbb{P}_k^1, K \otimes \bar{\mathcal{L}}_\psi(yx))$ is canonically identified with the fibre over y of $\mathcal{F}_\psi(K|_{\mathbb{A}_k^1})$, so that the deformation above is nothing but $\mathcal{F}_\psi(K|_{\mathbb{A}_k^1})$. In particular, by the fact of involutivity of \mathcal{F}_ψ , to consider this deformation is to consider $K|_{\mathbb{A}_k^1}$.

Let \mathcal{L}_ψ be the smooth $\bar{\mathbb{Q}}_\ell$ -sheaf of Artin-Schreier of rank 1 over $\mathbb{G}_{a, \mathbb{F}_p}$ associated to the character $\psi : \mathbb{F}_p \rightarrow \bar{\mathbb{Q}}_\ell^\times$. Let S a sheaf of finite type over k and $E \xrightarrow{\pi} S$, $\langle \cdot, \cdot \rangle : E \times_S E' \rightarrow \mathbb{G}_{a, k}$ the canonical paring and $pr : E \times_S E' \rightarrow E$, $pr' : E \times_S E' \rightarrow E'$ the two canonical projections.

Definition 4.3 (Fourier-Deligne transformation). *The Fourier-Deligne transformation for $E \xrightarrow{\pi} S$, associated to ψ , is the triangulated functor*

$$\mathcal{F}_\psi : D_c^b(E, \bar{\mathbb{Q}}_\ell) \rightarrow D_c^b(E', \bar{\mathbb{Q}}_\ell)$$

defined by

$$\mathcal{F}_\psi(K) = Rpr'_!(pr^* K \otimes \mathcal{L}_\psi(\langle \cdot, \cdot \rangle))[r].$$

If $k = \mathbb{F}_q$, one denote by

$$\mathcal{C}(E(\mathbb{F}_q), \bar{\mathbb{Q}}_\ell) \xrightarrow{\hat{t}} \mathcal{C}(E(\mathbb{F}_q), \bar{\mathbb{Q}}_\ell)$$

the Fourier transformation for functions defined by

$$\hat{t}(e') = \sum_{\pi(e) = \pi'(e'), e \in E(\mathbb{F}_q)} t(e) \psi_q(\langle e, e' \rangle), \forall e' \in E'(\mathbb{F}_q).$$

Theorem 4.3. ([8]) If $k = \mathbb{F}_q$, $K \in D_c^b(E, \bar{\mathbb{Q}}_\ell)$, one has

$$t_{\mathcal{F}_\psi(K)} = (-1)^r \hat{t}_K.$$

Theorem 4.4. ([8]) Denoting by \mathcal{F}' the Fourier-Deligne transformation associated to ψ for the vector bundle $E' \xrightarrow{\pi'} S$. Then one has a functorial isomorphism

$$\mathcal{F}' \circ \mathcal{F}(K) \cong a_* K(-r)$$

for $K \in \text{ob} D_c^b(E, \bar{\mathbb{Q}}_\ell)$

Definition 4.4. The convolution product for $E \xrightarrow{\pi} S$ is an internal operation

$$* : D_c^b(E, \bar{\mathbb{Q}}_\ell) \times D_c^b(E, \bar{\mathbb{Q}}_\ell) \longrightarrow D_c^b(E, \bar{\mathbb{Q}}_\ell)$$

defined by

$$K_1 * K_2 = R s_! (K_1 \boxed{\times}_S K_2).$$

Proposition 4.2. ([10]) One has a functorial isomorphism

$$\mathcal{F}(K_1 * K_2) = \mathcal{F}(K_1) \otimes \mathcal{F}(K_2)[-r]$$

for $(K_1, K_2) \in \text{ob}(D_c^b(E, \bar{\mathbb{Q}}_\ell) \times D_c^b(E, \bar{\mathbb{Q}}_\ell))$.

The next is the central theorem of Fourier-Deligne transformation.

Theorem 4.5. ([8])

For every $K \in \text{ob} D_c^b(E, \bar{\mathcal{F}}_\ell)$, the morphism for forgetting supports

$$Rpr'_!(pr^* K \otimes \mathcal{L}(\langle \ , \ \rangle)) \longrightarrow Rpr'_*(pr^* K \otimes \mathcal{L}(\langle \ , \ \rangle))$$

is an isomorphism.

Theorem 4.6. ([8]) \mathcal{F} transforms the perverse $\bar{\mathbb{Q}}_\ell$ -sheaves over E into those over E' . The functor

$$\mathcal{F} : \text{Perv}(E, \bar{\mathbb{Q}}_\ell) \longrightarrow \text{Perv}(E', \bar{\mathbb{Q}}_\ell)$$

is equivalence between abelian categories with $\mathcal{F}'(-)(r)$ a quasi-inverse.

Further, \mathcal{F} transforms the simple perverse $\bar{\mathbb{Q}}_\ell$ -sheaves over E into the simple perverse $\bar{\mathbb{Q}}_\ell$ -sheaves over E' .

4.1 Complex case

Let $p : E \rightarrow X$ a complex vector bundle of rank r over X and $p^* : E' \rightarrow X$ its dual bundle. One denotes by $D_- = \{(e, e^*) \in E \times_X E' \mid \text{Re}\langle e, e^* \rangle \leq 0\}$, $q : D_- \rightarrow E, q' : D_- \rightarrow E'$.

Definition 4.5. ([2],[3],[11], [13]) For a complex G of $D^b(\mathbb{C}_E)$ we define $\mathcal{F}(G) := \mathcal{R}q'_! q^{-1} G$, which is said to be geometric Fourier (Brylinski-Malgrange-Verdier or Fourier-Sato) transformation of G .

If in particular X reduces to a point, E, E' are complex vector spaces of dimension r . Then one has formal Fourier transformation between the quasi-coherent D_E -modules and the quasi-coherent $D_{E'}$ -modules:

$$\mathcal{F} : \Gamma(E', D_{E'}) \rightarrow \Gamma(E, D_E)$$

$\mathcal{F}(\xi_i) = \partial_{x_i}$ for $i = 1, \dots, r$, $\mathcal{F}(\partial_{\xi_i}) = -x_i$ for $i = 1, \dots, r$ if (x_i) for $i = 1, \dots, r$ is a system of regular parameters on E and (ξ_i) for $i = 1, \dots, r$ is a system of regular parameters on E' . Let $\mathbb{A} = \text{Spec}(\mathbb{C}[t])$ and L a $D_{\mathbb{A}}$ -module given by the $\mathcal{O}_{\mathbb{A}}$ with the canonical connection.

Proposition 4.3. ([2],[3],[11], [13]) Let M be a quasi-coherent D_E -module. Then there exists an isomorphism

$$\mathcal{F}(M) \cong \int_{q'_*} q^* M \otimes_{\mathcal{O}_{E \times E'}} \langle \ , \ \rangle^* L$$

Definition 4.6. ([2],[3],[11], [13])

$$\mathcal{F}_!(M) := \int_{q^!} q^* M \otimes_{\mathcal{O}_{\mathcal{E} \times \mathcal{E}'}} \langle \ , \ \rangle^* L$$

Theorem 4.7. ([2],[3],[11], [13]) If M is a holonomic D_E -module, the natural homomorphism

$$\mathcal{F}(M) \rightarrow \mathcal{F}_!(\mathcal{M})$$

Let $Eu = \sum_{i=1}^r x_i \partial_{x_i}$ be Euler vector field.

Theorem 4.8 (Malgrange([11])). Let M be a regular holonomic D_E -module such that the action of Eu on M is locally finite. Then $\mathcal{F}(\mathcal{M})$ is a regular holonomic $D_{E'}$ -module and the geometric Fourier transformation commutes with de Rham functor with respect to M .

We propose the following definition.

Definition 4.7. Let $\mathbb{A} = \text{Spec}(\mathbb{C}[t])$. One denotes by $\mathbb{C}_{\mathbb{A}}(1) = \mathbb{C}_{\mathbb{A}} \otimes x \partial_x$

Proposition 4.4 (Stabilities). 1. The category of q -pure \mathbb{C} -sheaves of weight n is stable by the operations sub-quotients, by extension, by inverse image and by direct image of finite morphism.

2. The tensor product of q -pure \mathbb{C} -sheaves of weight n and m is q -pure \mathbb{C} -sheaf of weight $n+m$. The dual of smooth q -pure \mathbb{C} -sheaf of weight n is q -pure of weight $-n$.

3. Note that $\mathbb{C}(1)$ is q -pure of weight -2 .

Proof. Obvious. □

We recall Laumon's argument.

Lemma 4.3. ([8])

Let $X, j : U \hookrightarrow X$ and S as above. Fix a point $s \in S$. If F is a smooth ι -pure $\bar{\mathbb{Q}}_{\ell}$ -sheaf with ι -weight $w \in \mathbb{R}$ over U such that the action of I_s over $F_{\eta_s}^-$ is unipotent of rank 2, i.e., I_s acts trivially over the quotient

$$F_{\eta_s}^- / F_{\eta_s}^{-I_s},$$

then for each proper value α of Frob_s acts over this quotient one has

$$|\iota(\alpha)| = q^{\deg(s)(w+1)/2}$$

Lemma 4.4. ([9], [8])

If F is a smooth ι -pure $\bar{\mathbb{Q}}_{\ell}$ -sheaf with $w \in \mathbb{Z}$, then for each $x \in |X|$, one has

$$\iota \det(1 - t \text{Frob}_x, j_*(F \oplus F^{\vee}(-w))) \in \mathbb{R}[t].$$

Theorem 4.9. ([8])

Purity theorem over a curve \iff Purity theorem over a vector line

Theorem 4.10 (Laumon's purity theorem over a vector line([8])). Let $A = \text{Spec}(\mathbb{F}_q[x])$ be a vector line over \mathbb{F}_q , $j : U \hookrightarrow A$ a dense open and a smooth ι -pure $\bar{\mathbb{Q}}_{\ell}$ -sheaf with $w \in \mathbb{Z}$ over U . Suppose, further, that F is irreducible, unramified at the infinite point and non geometrically constant. Then for every proper value α of Frob_q acts over $H_c^1(A \otimes_{\mathbb{F}_q} \bar{k}, F)$, one has the estimation

$$|\iota(\alpha)| \leq q^{(w+1)/2}.$$

Proof. [Laumon purity] Consider the Fourier-Deligne transformation \mathcal{F} of $A = \text{Spec}(\mathbb{F}_q[x])$ to $A' = \text{Spec}(\mathbb{F}_q[x'])$ relative to a fixed character ψ . Fourier-Deligne transformation exchanging the tree types, one has

$$\mathcal{F}(j_* F[1]) = j'_* F'[1],$$

where $j' : A' - \{0\} \hookrightarrow A'$ is an inclusion and F' is a smooth irreducible \mathcal{Q}_ℓ -sheaf over $A' - \{0\}$. Since for perverse K' ,

$$0 \rightarrow \mathcal{H}^{-1}(K'_{\bar{s}}) \rightarrow F'_{\eta_{s'}} \rightarrow R^{-1}\Phi_{\eta_{s'}}^-(K') \rightarrow \mathcal{H}^0(K'_{\bar{s}}) \rightarrow 0,$$

one has

$$0 \rightarrow H_c^1(A \otimes_{\mathbb{F}_q} \bar{k}, j_* F) \rightarrow F'_{\eta_{0'}} \rightarrow F_{\infty}^-(-1) \rightarrow 0.$$

The inertia group $I_{0'}$ acts trivially on $H_c^1(A \otimes_{\mathbb{F}_q} \bar{k}, j_* F)$ and $F_{\infty}^-(-1)$. Thus $G_{0'}$ acts through $\text{Gal}(\bar{k}/\mathbb{F}_q)$ over these spaces. One has

$$(F'_{\eta_{0'}})^{I_{0'}} = H_c^1(A \otimes_{\mathbb{F}_q} \bar{k}, j_* F)$$

and

$$\mathcal{F}^{(\infty, 0')}(F|_{\eta_{\infty}}) = F_{\infty}^- \otimes \mathcal{F}^{(\infty, 0')}(\bar{\mathbb{Q}}_\ell) = F_{\infty}^-(-1).$$

$\mathcal{F}^{(\infty, 0')}$ is the local Fourier transformation of $(D_{\infty}, 1/x)$ to (D'_{∞}, x') . Later we show that F' is ι -pure with a certain weight $w' \in \mathbb{R}$. Making use of it, one can prove the theorem as follows:

$$F'_{\eta_{0'}} / (F'_{\eta_{0'}})^{I_{0'}} = F_{\eta_{0'}}^-(-1)$$

is ι -pure with $w' + 1$. On the other hand, $F'_{\eta_{0'}}$ is ι -pure with ι -weight w , hence

$$w' = w + 1.$$

Therefore, we complete the proof. Next, we show F' is ι -pure.

$$\mathcal{F}_{\psi}(j_*(F \oplus F^{\vee}(-w)))[-1]) \oplus \mathcal{F}_{\psi^{-1}}(j_*(F \oplus F^{\vee}(-w)))[-1])$$

This perverse $\bar{\mathbb{Q}}_\ell$ -sheaf is of the form

$$j'_* G'[1]$$

for a smooth $\bar{\mathbb{Q}}_\ell$ -sheaf G' over $A' - \{0'\}$. Since $j'_* F'[1] \hookrightarrow j'_* G'[1]$, F' is a direct factor of G' over $A' - \{0\}$. Thus it suffices to prove that G' is ι -real. By formula of function L due to Grothendieck,

$$\iota \det(1 - t \text{Frob}_{x'}, G') = \prod_{x \in |A \otimes_{\mathbb{F}_q} k(x')|} \frac{1}{Q_{x, x'}^{\psi}(t^{\deg(x)}) Q_{x, x'}^{\psi^{-1}}(t^{\deg(x)})},$$

where

$$Q_{x, x'}^{\psi}(t) = P_x(t \psi(\text{tr}_{k(x)/\mathbb{F}_p}(x \cdot x')))$$

and

$$P_x(t) = \iota \det(1 - t \text{Frob}_x, j_*(F \oplus F^{\vee}(-w))).$$

Note that $k(x) \supset k(x') \supset \mathbb{F}_p$. Since $P_x(t) \in \mathbb{R}[t]$, for any $x \in |A|$, one has

$$\iota \det(1 - t \text{Frob}_{x'}, G') \in \mathbb{R}[t],$$

for any $x' \in |A'| - \{0\}$. This accomplishes the proof. □

5 Complex variety case

One obtains the following equivalent statement by direct interpretation.

Theorem 5.1.

$$\text{Purity theorem over a curve} \iff \text{Purity theorem over a vector line}$$

Hence we have only to prove the following purity theorem over a vector line.

Theorem 5.2. *Let $A = \text{Spec}(\mathbb{C}[x])$ be a vector line over \mathbb{C} , $j : U \hookrightarrow A$ a dense open and a smooth q -pure \mathbb{C} -sheaf with $w \in \mathbb{Z}$ over U . Suppose, further, that F is irreducible, unramified at the infinite point and non geometrically constant. Then for every proper value α of Frob_q acts over $H_c^1(A \otimes_K \bar{K}, F)$, one has the estimation*

$$|\iota(\alpha)| \leq q^{(w+1)/2}.$$

Proof. Consider the geometric Fourier transformation \mathcal{F} of $A = \text{Spec}(\mathbb{C}[x])$ to $A' = \text{Spec}(\mathbb{C}[x'])$. The geometric Fourier transformation exchanging the tree types, one has

$$\mathcal{F}(j_* F[1]) = j'_* F'[1],$$

where $j' : A' - \{0\} \hookrightarrow A'$ is an inclusion and F' is a smooth irreducible \mathbb{C} -sheaf over $A' - \{0\}$. Since for perverse K' ,

$$0 \rightarrow \mathcal{H}^{-1}(K'_{\bar{s}'}) \rightarrow F'_{\bar{\eta}_{s'}} \rightarrow R^{-1}\Phi_{\bar{\eta}_{s'}}^-(K') \rightarrow \mathcal{H}^0(K'_{\bar{s}'}) \rightarrow 0,$$

one has

$$0 \rightarrow H_c^1(A \otimes_K \bar{K}, j_* F) \rightarrow F'_{\bar{\eta}_{0'}} \rightarrow F'_{\infty}(-1) \rightarrow 0.$$

The inertia group $I_{0'}$ acts trivially on $H_c^1(A \otimes_K \bar{K}, j_* F)$ and $F'_{\infty}(-1)$. Thus $G_{0'}$ acts through $\text{Gal}(\bar{K}/K)$ over these spaces. One has

$$(F'_{\bar{\eta}_{0'}})^{I_{0'}} = H_c^1(A \otimes_K \bar{K}, j_* F)$$

and

$$\mathcal{F}^{(\infty, 0')}(\mathcal{F}|_{\eta_{\infty}}) = \mathcal{F}_{\infty} \otimes \mathcal{F}^{(\infty, 0')}(\mathbb{C}) = \mathcal{F}_{\infty}(-\infty)$$

$\mathcal{F}^{(\infty, 0')}$ is the local Fourier transformation of $(D_{\infty}, 1/x)$ to (D'_{∞}, x') . Later we show that F' is q -pure with a certain weight $w' \in \mathbb{R}$. Making use of it, one can prove the theorem as follows:

$$F'_{\bar{\eta}_{0'}} / (F'_{\bar{\eta}_{0'}})^{I_{0'}} = F'_{\bar{\eta}_{0'}}(-1)$$

is q -pure with $w' + 1$. On the other hand, $F'_{\bar{\eta}_{0'}}$ is q -pure with weight w , hence

$$w' = w + 1.$$

Next, we show F' is q -pure. $\mathcal{F}(j_* \mathcal{F}[\infty])$ is a direct factor of

$$\mathcal{F}(j_*(F \oplus F^{\vee}(-w))[-1])$$

This perverse \mathbb{C} -sheaf is of the form

$$j'_* G'[1]$$

for a smooth \mathbb{C} -sheaf G' over $A' - \{0'\}$. Since $j'_* F'[1] \hookrightarrow j'_* G'[1]$, F' is a direct factor of G' over $A' - \{0\}$. Thus it suffices to prove that G' is real. By formula of function L due to Grothendieck,

$$\det(1 - t\text{Frob}_{x'}, G') = \prod_{x \in |A \otimes_K K(x')|} \frac{1}{Q_{x, x'}(t^{\deg(x)})},$$

where

$$Q_{x, x'}(t) = P_x(\text{ttr}_{K(x)/K}(x \cdot x'))$$

and

$$P_x(t) = \det(1 - t\text{Frob}_x, j_*(F \oplus F^{\vee}(-w))).$$

Note that $K(x) \supset K(x') \supset K$. Since $P_x(t) \in \mathbb{R}[t]$, for any $x \in |A|$, one has

$$\det(1 - t\text{Frob}_{x'}, G') \in \mathbb{R}[t],$$

for any $x' \in |A'| - \{0\}$. This completes the proof. \square

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