Stretching Rays at Critically Prefixed Real Cubic Polynomials

Shizuo Nakane *
Tokyo Institute of Polytechnics
1583 Iiyama, Atsugi, Kanagawa 243-0297, Japan

In this note, the dynamics of real cubic polynomials is considered. Especially, in the parameter space, landing of stretching rays at critically prefixed polynomials is investigated. It turns out that an interval worth of stretching rays land at such points.

1 Introduction

In this note, we shall investigate the dynamics of a family of real cubic polynomials :

$$P(z) = P_{A,B}(z) = z^3 - 3Az + \sqrt{B},\ A, B > 0.$$

Since the stretching ray passing through a point in this family stays in this family, we call such stretching rays *real*. Our main concern is on the landing of real stretching rays for this family, especially on $Preper_{(1)1}$, the locus where one critical value becomes a fixed point. In the first quadrant, the boundary of the connectedness locus is very simple. It consists of two real algebraic curves. And stretching rays must accumulate on these curves. See Figure 1. This really simplifies things. Our main result is that an interval worth of real stretching rays land at a certain class of critically prefixed maps.

There are few works on the landing of stretching rays. Kiwi [Ki] has considered critical portraits for polynomials in the visible shift locus of arbitrary degree and has investigated the relation between their combinatorics and those of their impressions. Especially, if the critical portrait consists of strictly preperiodic angles, its impression consists of a single polynomial, whose critical points are strictly preperiodic. Consequently, stretching rays through polynomials with such critical portrait land at a critically preperiodic polynomial. Willumsen [W] considered the accumulation of stretching rays on $Per_1(1)$, the locus where P has a parabolic fixed point with multiplier 1, in the family of complex cubic polynomials.

Here we consider stretching rays only in the family of real cubic polynomials. Although Kiwi [Ki] has obtained a deep result in more general settings, our result is not contained in his. In fact, our real stretching rays are not contained in the visible shift locus, where he considered. Besides, we give a more elementary proof.

2 Stretching rays

Let $\mathcal{P}_d$ be the family of monic centered polynomials of degree $d \geq 2$. For $P \in \mathcal{P}_d$, let $K(P)$ be its *filled-in Julia set*, that is, the set of points $z \in \mathbf{C}$ whose orbit $\{P^n(z); n \geq 0\}$ is bounded and let $J(P)$ be its *Julia set*, the boundary of $K(P)$. The *connectedness locus* $\mathcal{C}_d$ or the *escape locus* $\mathcal{E}_d$ of $\mathcal{P}_d$ is the set of $P \in \mathcal{P}_d$ whose Julia set $J(P)$ is connected or disconnected respectively. Let φ_P be its *Böttcher coordinate* defined in a neighborhood of ∞. It satisfies $\varphi_P(P(z)) = \varphi_P(z)^d$ and tangent to identity at ∞. Let $h_P(z) = \log_+ |\varphi_P(z)|$ be the *Green function* for P, which is continued continuously to the whole plane by the functional equation $h_P(P(z)) = d \log_+ |\varphi_P(z)|$ and is harmonic in $\mathbf{C} - K(P)$. Put $G(P) = \max\{h_P(\omega); \omega$ is a critical point of $P\}$. Then φ_P can be continued analytically to $U_P =$

* Tokyo Institute of Polytechnics
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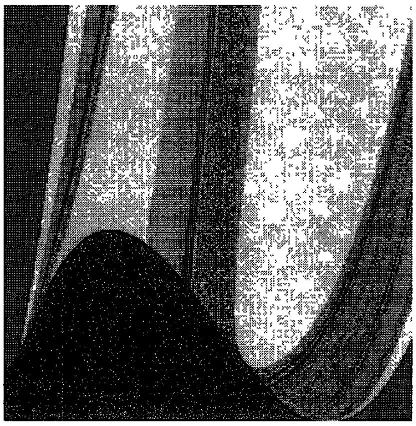

Figure 1: The connectedness locus $\mathcal{C}_3^R$

$\{z; h_P(z) > G(P)\}$. For a complex number $u \in H_+ = \{u = s + it \in \mathbf{C}, s > 0\}$, put $f_u(z) = z|z|^{u-1}$ and we define a P-invariant almost complex structure σ_u by

$$\sigma_u = \begin{cases} (f_u \circ \varphi_P)^* \sigma_0 \text{ on } U_P, \\ \sigma_0 \text{ on } K(P). \end{cases}$$

Then, by the Measurable Riemann Mapping Theorem, σ_u is integrated by an appropriately normalized qc-map F_u so that $P_u = F_u \circ P \circ F_u^{-1} \in \mathcal{P}_d$. Since the same theorem says F_u depends holomorphically on u, so does P_u. Thus we define a holomorphic map $W_P : H_+ \to \mathcal{P}_d$ by $W_P(u) = P_u$. The Böttcher coordinate φ_{P_u} of P_u is equal to $f_u \circ \varphi_P \circ F_u^{-1}$. This operation is called *wringing*. Since P_u is hybrid equivalent to P, it holds $P_u \equiv P$ for $P \in \mathcal{C}_d$. For $P \in \mathcal{E}_d$, we define the *stretching ray* through P by

$$R(P) = W_P(\mathbf{R}_+) = \{P_s; s \in \mathbf{R}_+\}.$$

For example, in case $d = 2$, stretching rays coincide with the external rays for the Mandelbrot set. As for stretching rays, see Branner [Br] or Branner-Hubbard [BH2]. The following is a direct consequence from the definition.

Lemma 2.1 *Let ω_j, for $j = 1, 2$ be two escaping critical points of $P \in \mathcal{E}_d$. Then $\tilde{\eta}(P_s) = \dfrac{h_{P_s}(\omega_1)}{h_{P_s}(\omega_2)}$ is invariant on the streching ray $R(P)$ through P.*

proof. Since $|\varphi_{P_s}(z)| = |f_s \circ \varphi_P \circ F_s^{-1}(z)| = |\varphi_P \circ F_s^{-1}(z)|^s$, we have $h_{P_s}(z) = s \cdot h_P(F_s^{-1}(z))$ and

$$\tilde{\eta}(P_s) = \frac{h_{P_s}(F_s(\omega_1))}{h_{P_s}(F_s(\omega_2))} = \frac{h_P(\omega_1)}{h_P(\omega_2)} = \tilde{\eta}(P).$$

This completes the proof. □

Generally speaking, in this lemma, we cannot replace $h_P(\omega_j) = \log|\varphi_P(\omega_j)|$ by $\log \varphi_P(\omega_j)$ in the definition of $\tilde{\eta}(P)$. But, in case of real cubic polynomials in the first quadrant, we can do so since both critical points $\pm\sqrt{A}$ are real and their orbits lie on the positive real axis in the Böttcher coordinate. Here is an advantage of considering the real cubic polynomials. We set $\zeta_P(z) = \dfrac{\log\log\varphi_P(z)}{\log 3}$ and define, for $P \in \mathcal{E}_3^2$ (the real *shift locus*, i.e. the locus where both critical points escape), the *Böttcher vector* $\eta(P)$ by

$$\begin{aligned} \eta(P) &= \frac{\log h_P(\sqrt{A}) - \log h_P(-\sqrt{A})}{\log 3} \\ &= \zeta_P(\sqrt{A}) - \zeta_P(-\sqrt{A}). \end{aligned}$$

Note that, since $\varphi_P(\pm\sqrt{A}) > 1$, $\zeta_P(\pm\sqrt{A})$ is well defined. Then Lemma 2.1 implies the following.

Lemma 2.2 *On the stretching ray $R(P)$ through $P \in \mathcal{E}_3^2$, $\eta(P_s)$ is invariant.*

This lemma will play an important role in the following sections.

3 The parameter space of real cubic polynomials

We restrict our attention to the first quadrant of the parameter space of real cubic polynomials. Then both critical points $\pm\sqrt{A}$ of P are real.

Lemma 3.1 *(Milnor [M1]) The real connectedness locus $\mathcal{C}_3^R$ is bounded by two real algebraic curves :*

$$\begin{aligned} Per_1(1) &= \{B = 4(A+1/3)^3; 0 \le A \le 1/9\}, \\ Preper_{(1)1} &= \{B = 4A(A-1)^2; 1/9 \le A \le 1\}. \end{aligned}$$

In the region $B < 4(A+1/3)^3$, P has three distinct real fixed points. On $Per_1(1)$, two of them collapse into a parabolic fixed point of multiplier 1. And in the region $B > 4(A+1/3)^3$, P has only one real fixed point. Now let β_P and β'_P be the real fixed points of P at which the external rays with angles 0 and 1/2 land respectively. And we denote the other one by β''_P. In the region $B > 4(A+1/3)^3$, only β'_P exists. In the region $B \le 4(A+1/3)^3$, β_P and β'_P are the maximum and minimum real fixed points respectively. $Per_1(1)$ is the set of parameters such that $\beta_P = \beta''_P = \sqrt{A+1/3}$ is the parabolic fixed point with multiplier 1. $Preper_{(1)1}$ is the set of parameters satisfying $P(-\sqrt{A}) = \beta_P$.

Figure 1 is the parameter space of our family. The black region is the connectedness locus. Its complement is gradated in order to emphasize stretching rays. That is, $\mathcal{E}_3^2$ is gradated by the Böttcher vector. The locus $\mathcal{E}_3^1$ where only one critical point escapes is gradated by the period of the attracting cycle.

4 Landing of stretching rays on critically prefixed polynomials

In this section, we consider the landing of stretching rays in the real shift locus $\mathcal{E}_3^2$. More precisely, we consider the region $\{(A,B) \in \mathcal{E}_3^2; B < 4(A+1/3)^3\}$. Note that this region is disjoint from the visible shift locus. Roughly speaking, a map in the shift locus is *visible* if each critical point is the terminating point of an external radius and its external angle is well defined. As for a precise definition, see Kiwi [Ki]. Figure 2 is the Julia set of a map in the real shift locus. It gives a ternary decomposition of the complement of the filled-in Julia set. And we can see some external rays and critical points. At the critical point $-\sqrt{A}$, external rays of angles 1/3 and 2/3 terminate. So, it is visible. But the other critical point $\sqrt{A}$ is not visible. Since both critical points are real in the first quadrant, we consider the dynamics only on the real axis.

Lemma 4.1 *In the region $4A(A-1)^2 \le B \le 4(A+1/3)^3$, the orbit of a point $x > \beta'_P$ escapes to ∞ if and only if there exists $k \ge 0$ such that $P^k(x) > \beta_P$. Especially, if $P(-\sqrt{A}) \le \beta_P$, it follows $[\beta'_P, \beta_P] = K(P) \cap \mathbf{R}$.*

proof. Since $4A(A-1)^2 \le B$ implies $P(\sqrt{A}) \ge \beta'_P$, we have $P(x) \ge \beta'_P$ for $x \ge \beta'_P$. Hence the first statement follows. Furthermore, if $P(-\sqrt{A}) \le \beta_P$, it follows $P([\beta'_P, \beta_P]) \subset [\beta'_P, \beta_P]$ and the last conclusion holds. □

Corollary 4.1 *In the region $4A(A-1)^2 \le B \le 4(A+1/3)^3$, $\sqrt{A}$ escapes if and only if there exists $k \ge 0$ such that $P^k(\sqrt{A}) > \beta_P$. In this case $P^n(\sqrt{A}) \to +\infty$. In the region $B < 4A(A-1)^2$, $A > 1$, $P(\sqrt{A}) < \beta'_P$ and $P^n(\sqrt{A}) \to -\infty$. In the region $B > 4(A+1/3)^3$, $P^n(\sqrt{A}) \to +\infty$.*

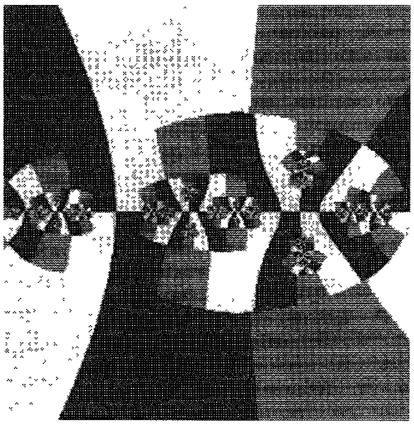

Figure 2: The Julia set of a map in the real shift locus

Lemma 4.2 *In the first quadrant, $G(P) = h_P(-\sqrt{A})$. That is, the critical point $-\sqrt{A}$ escapes faster than the other critical point $\sqrt{A}$.*

proof. Since $P(-\sqrt{A}) > P(\sqrt{A}) > \sqrt{A}$ in $B > 4(A+1/3)^3$, we have $P^k(-\sqrt{A}) > P^k(\sqrt{A})$ for any k and the conclusion follows. In the region $4A(A-1)^2 \leq B \leq 4(A+1/3)^3$, the conclusion follows from Lemma 4.1. In the region $B \leq 4A(A-1)^2$, $A > 1$, $|P^k(-\sqrt{A})| > |P^k(\sqrt{A})|$ holds for any k. This completes the proof. □
Then, from Corollary 4.1 and Lemma 4.2, it follows

$$\mathcal{E}_3^1 = \{(A,B) \in \mathbf{R}_+^2 - \mathcal{C}_3^R; P^k(\sqrt{A}) \leq \beta_P \text{ for any } k \geq 0\},$$

and, for any connected component U of $\mathcal{E}_3^2 \cap \{4A(A-1)^2 < B < 4(A+1/3)^3\}$, there exists $k \geq 1$ such that

$$P^j(\sqrt{A}) < \beta_P,\ 0 \leq j \leq k,\ P^{k+1}(\sqrt{A}) > \beta_P,$$

in U. Hence its boundary ∂U is contained in the real algebraic set $P^{k+1}(\sqrt{A}) = \beta_P$.

Lemma 4.3 *Each connected component R_{k+1} of $\{(A,B) \in \mathcal{E}_3^1; P^{k+1}(\sqrt{A}) = \beta_P\}$ forms a stretching ray. It lands at a point $(A,B) \in Preper_{(1)1}$ satisfying $P^k(\sqrt{A}) = -\sqrt{A}$.*

proof. It is easy to see R_{k+1} forms a stretching ray. Since both $Preper_{(1)1}$ and R_{k+1} are real algebraic curves, either they coincide or their intersection set is locally finite. Since they do not coincide, R_{k+1} must land at a point $(A_0, B_0) \in Preper_{(1)1}$ and we have $P_0^{k+1}(\sqrt{A_0}) = \beta_{P_0} = P_0(-\sqrt{A_0})$. Hence $P_0^k(\sqrt{A_0})$ equals to $-\sqrt{A_0}$ or β_{P_0}. Suppose $P_0^k(\sqrt{A_0}) = \beta_{P_0}$. Then there exists a $j < k$ such that $P_0^j(\sqrt{A_0}) = -\sqrt{A_0}$. Take such minimum j. Then the following Lemma 4.4 implies $(\partial/\partial B)\{P_{A,B}^{j+1}(\sqrt{A}) - \beta_P\} > 0$ at (A_0, B_0). By the implicit function theorem, there is a real analytic curve $R_{j+1} : P_{A,B}^{j+1}(\sqrt{A}) = \beta_P$ through (A_0, B_0). Since $P(-\sqrt{A}) \geq \beta_P$ on this curve, this curve yields two stretching rays. By Lemma 4.4 and the fact that P^{k-j} is monotone increasing at β_P, $P^{j+1}(\sqrt{A}) > \beta_P$ if and only if $P^{k+1}(\sqrt{A}) > \beta_P$. Hence R_{k+1} coincides with R_{j+1}. This contradicts the definition of R_{k+1}. Thus $P_0^k(\sqrt{A_0}) = -\sqrt{A_0}$. This completes the proof. □

Lemma 4.4 *Suppose $P_0 = P_{A_0,B_0} \in Preper_{(1)1}$ satisfies $P_0^k(\sqrt{A_0}) = -\sqrt{A_0}$.*
Then $(\partial/\partial B)(P^{k+1}(\sqrt{A}) - \beta_P) > 0$ at (A_0, B_0).

proof. Put $P_\epsilon = P_{A_0,B_0+\epsilon}$. It easily follows that, for fixed A, β_P is a monotone decreasing function of B. Hence we have only to show $P_\epsilon^{k+1}(\sqrt{A_0})$ is a monotone increasing function of $\epsilon > 0$. From the

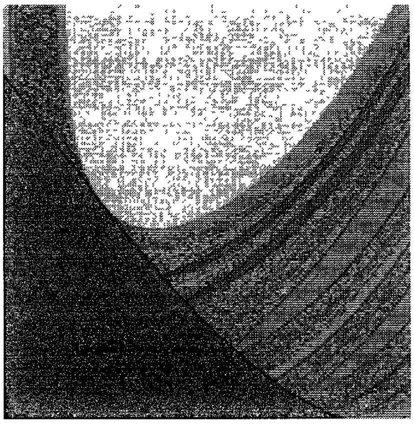

Figure 3: Stretching rays tangent to $Preper_{(1)1}$

assumption, it follows

$$\begin{aligned}(\partial/\partial\epsilon)P_\epsilon^{k+1}(\sqrt{A_0}) &= (\partial/\partial\epsilon)P_\epsilon(P_\epsilon^k(\sqrt{A_0}))\\ &= \frac{1}{2\sqrt{B_0+\epsilon}} + P_\epsilon'(P_\epsilon^k(\sqrt{A_0}))(\partial/\partial\epsilon)P_\epsilon^k(\sqrt{A_0})\\ &= \frac{1}{2\sqrt{B_0}} + O(\epsilon) > 0.\end{aligned}$$

This completes the proof. □
The same estimate holds also for $(A, B) \in Preper_{(1)1}$ close to (A_0, B_0). This proof says that R_{k+1} is continued, as a real analytic curve, to a neighborhood of $A = A_0$, which we also denote by R_{k+1}.

Lemma 4.5 *R_{k+1} is tangent to $Preper_{(1)1}$ at (A_0, B_0).*

proof. Put $h(A) = P^{k+2}(\sqrt{A}) - P^{k+1}(\sqrt{A})$. Then we have

$$\begin{aligned}h &= P^2(P^k(\sqrt{A})) - P^2(-\sqrt{A}) - \{P(P^k(\sqrt{A})) - P(-\sqrt{A})\} + P^2(-\sqrt{A}) - P(-\sqrt{A})\\ &= (P^2-P)''(-\sqrt{A})(P^k(\sqrt{A}) + \sqrt{A})^2/2 + O((P^k(\sqrt{A}) + \sqrt{A})^3) + P^2(-\sqrt{A}) - P(-\sqrt{A}).\end{aligned}$$

Since $h = 0$ on R_{k+1} and

$$(P^2 - P)''(-\sqrt{A}) = \{P'(P(-\sqrt{A})) - 1\}P''(-\sqrt{A}) = 6\sqrt{A}(1 - 9A) < 0,$$

it follows

$$P^2(-\sqrt{A}) - P(-\sqrt{A}) = -(P^2 - P)''(-\sqrt{A})(P^k(\sqrt{A}) + \sqrt{A})^2/2 + O((P^k(\sqrt{A}) + \sqrt{A})^3) \geq 0.$$

Thus the component of the set $P^{k+1}(\sqrt{A}) = \beta_P$ sits on one side of $Preper_{(1)1}$ and it is real analytic even at (A_0, B_0). Hence it must be tangent to $Preper_{(1)1}$ at (A_0, B_0). This completes the proof. □
Figure 3 is an enlargement of Figure 1. This supports the above lemma.

Lemma 4.6 *Any point $(A_0, B_0) \in Preper_{(1)1}$ satisfying $P_0^k(\sqrt{A_0}) = -\sqrt{A_0}$ is the landing point of just two stretching rays, which are of the form R_{k+1}.*

proof. From Lemma 4.4, it follows $P^{k+1}(\sqrt{A}) > \beta_P$ for $(A, B) = (A_0, B_0 + \epsilon)$. On the other hand, $P^{k+1}(\sqrt{A}) \leq \beta_P$ holds on $Preper_{(1)1}$. Now, by the intermediate value theorem, there exists a curve $P^{k+1}(\sqrt{A}) = \beta_P$ between $Preper_{(1)1}$ and $(A_0, B_0+\epsilon)$ for any $\epsilon > 0$. This curve must land at (A_0, B_0). Since $(P^{k+1}(\sqrt{A}) - \beta_P)|_{A=A_0}$ is a monotone increasing function of B at B_0, there are only two rays landing at (A_0, B_0). □

Corollary 4.2 *For the point (A_0, B_0) as above, $(A_0, B_0+\epsilon) \in \mathcal{E}_3^2$ for small $\epsilon > 0$. Especially, stretching rays of the form R_{k+1} are contained in the boundary of a connected component of $\mathcal{E}_3^2$.*

Stretching rays through points in the connected component of $\mathcal{E}_3^2$ bounded by two such rays must land also at (A_0, B_0).

Theorem 4.1 *For any $P \in \mathcal{E}_3^2 \cap \{4A(A-1)^2 < B < 4(A+1/3)^3\}$, there exists $k > 0$ such that $P^{k+1}(\sqrt{A}) > \beta_P$. Take such minimum k. Then the stretching ray $R(P)$ through P lands at a point $(A_0, B_0) \in Preper_{(1)1}$ satisfying $P_0^k(\sqrt{A_0}) = -\sqrt{A_0}$. The stretching ray $R(P)$ through $P \in \mathcal{E}_3^2 \cap \{B < 4A(A-1)^2\}$ lands at $(1, 0)$. Conversely, such a point $(A_0, B_0) \in Preper_{(1)1}$ is the landing point of an interval worth of stretching rays of the above property.*

proof. We have only to show that there exist an interval worth of stretching rays landing at (A_0, B_0). Let $U = U_k$ be the connected component of $\mathcal{E}_3^2 \cap \{4A(A-1)^2 < B < (A+1/3)^3\}$ containing P. For any $P \in U_k$, there exists $m \geq k+2$ such that $P^m(\sqrt{A}) > P(-\sqrt{A})$. Take such minimum m. If P approaches ∂U_k, m becomes arbitrarily large. Thus there are at least two stretching rays in U_k of the form $R_{k+1,m} : P^m(\sqrt{A}) = P(-\sqrt{A})$ for any $m \geq k+2$. They must land at (A_0, B_0). The Böttcher vector map $\eta(P)$ takes any values in $(-m, 1-m)$ in the region between two stretching rays $R_{k+1,m}$ and $R_{k+1,m+1}$. On the other hand, Lemma 2.2 says it is invariant on a stretching ray. Thus there are as many stretching rays as the Böttcher vectors landing at (A_0, B_0). This completes the proof. □

Lemma 4.7 *$R_{k+1,m}$ is real analytic and tangent to $Preper_{(1)1}$ at (A_0, B_0).*

proof. In order to show the real analyticity at (A_0, B_0), we have only to show

$$(\partial/\partial B)(P^m(\sqrt{A}) - P(-\sqrt{A})) = P'(P^{m-1}(\sqrt{A}))(\partial/\partial B)P^{m-1}(\sqrt{A}) > 0,$$

at (A_0, B_0). Since $P'(P^{m-1}(\sqrt{A})) > 0$ on $R_{k+1,m}$, it is sufficient to prove $(\partial/\partial B)P^{m-1}(\sqrt{A}) > 0$. The case $m = k+2$ follows from Lemma 4.4. For $m > k+2$, we have

$$\begin{aligned}(\partial/\partial B)P^{m-1}(\sqrt{A}) &= \frac{1}{2\sqrt{B}} + P'(P^{m-2}(\sqrt{A}))(\partial/\partial B)P^{m-2}(\sqrt{A}) \\ &> P'(P^{m-2}(\sqrt{A}))(\partial/\partial B)P^{m-2}(\sqrt{A}) \\ &> \prod_{j=k+1}^{m-2} P'(P^j(\sqrt{A})) \cdot (\partial/\partial B)P^{k+1}(\sqrt{A}) > 0.\end{aligned}$$

Thus we have shown the real analyticity of $R_{k+1,m}$ at (A_0, B_0). Next, since

$$\begin{aligned}0 &= P^m(\sqrt{A}) - P(-\sqrt{A}) = P(P^{m-1}(\sqrt{A})) - P(P(-\sqrt{A})) + P^2(-\sqrt{A}) - P(-\sqrt{A}) \\ &= P'(P(-\sqrt{A}))(P^{m-1}(\sqrt{A}) - P(-\sqrt{A})) + O((P^{m-1}(\sqrt{A}) - P(-\sqrt{A}))^2) \\ &\quad + P^2(-\sqrt{A}) - P(-\sqrt{A}),\end{aligned}$$

and $P^{m-1}(\sqrt{A}) < P(-\sqrt{A})$ on $R_{k+1,m}$, we have

$$\begin{aligned}P^2(-\sqrt{A}) - P(-\sqrt{A}) &= -P'(P(-\sqrt{A}))(P^{m-1}(\sqrt{A}) - P(-\sqrt{A})) \\ &\quad + O((P^{m-1}(\sqrt{A}) - P(-\sqrt{A}))^2) \geq 0\end{aligned}$$

for $(A, B) \in R_{k+1,m}$ close to (A_0, B_0). Hence the real analytic curve $P^m(\sqrt{A}) = P(-\sqrt{A})$ is tangent to $Preper_{(1)1}$ at (A_0, B_0). This completes the proof. □
Numerical experiments suggest the following.

Conjecture 4.1 *All the stretching rays but one in the region U_k as above land at (A_0, B_0) tangentially to $Preper_{(1)1}$. The exceptional one is the ray expressed by $P^k(\sqrt{A}) = -\sqrt{A}$, which lands transversally to $Preper_{(1)1}$. The Böttcher vector map η takes values exactly on $(-\infty, -k]$ in U_k.*

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