

# Deformation of Function Fields and Diophantine Problems

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In this article a theory of deformation of log varieties is founded by adopting the Serre dual tangential sheaf of Verdier dual of Deligne's differential sheaf with logarithmic poles. This theory is a key of Viehweg conjecture on variation of fibres of an algebraic fibre space. Another key is torsion freeness of higher direct sheaves of multiple relative dualizing sheaves. This is applied to some Diophantine problems.

## 1 Introduction

Our function fields are equivalent to the birational equivalence classes of complete algebraic varieties defined over the complex number field. Let  $f : X \rightarrow S$  be a proper smooth connected morphism between smooth varieties with relative divisors  $A, B$  on  $X/S$ . Assume that  $A + B$  is normal crossing. The dual notion of  $\Omega_{X/S}(A, B)$  is denoted by  $\Theta_{X/S}(A, B)$ . There exists a variation of deformation theory of Kodaira-Spencer-Kawamata's. If Kodaira-Spencer map  $\rho_{X/S}(A, B) : \Theta_S \rightarrow R^1 f_* \Theta_{X/S}(A, B)$  is zero, then  $X \setminus B$  is locally  $S$ -locally trivial outside  $A$ . This will be applied to Viehweg conjecture:  $\kappa(\det f_* \omega_{X/S}^{\otimes m}) \geq \text{var}(X/S)$  for any  $m > 0$  unless  $f_* \omega_{X/S}^{\otimes m} \neq 0$ . This statement gives deformation theory of function fields for algebraic varieties of non negative Kodaira dimension. The torsion freeness of  $R^1 f_* \omega_{X/S}^{\otimes m}$  takes a key-role. We apply these tools to Diophantine problems. We refer to the Flip conjectures, which will be written in detail elsewhere.

## 2 Diophantine Problems

### 2.1 Mordell-type Problem

**Theorem 2.1.** *Let  $k$  be a field of characteristic 0 and  $f : X \rightarrow S$  a surjective geometrically connected morphism of projective smooth varieties over  $k$ . Suppose that a general fibre of  $X/S$  is of general type. Then assuming the sections of  $X/S$  is dense in  $X$ , the variation of  $X/S$  is 0.*

*Proof.* One can reduce the base  $S$  to the curve  $C$  of genus more than 1. Let  $K$  denote the rational function field of  $C$ . Let  $\pi : \mathbf{P}(\Omega_{X_K/k}) \rightarrow X_K$  be the projective bundle of  $\Omega_{X_K/k}$  over  $X_K$  with the fundamental sheaf  $\mathcal{O}_P(1)$ . Let  $X_K$  be the generic fibre of  $f : X \rightarrow C$ . Let  $\pi_K : \mathbf{P}(\Omega_{X_K/k}) \rightarrow X_K$  be the restriction of  $\pi$  to  $X_K$ . By each canonical surjection  $\Omega_{X_K/k}|_{K_\lambda} \rightarrow \mathcal{O}_{K_\lambda}$  one has a rational point  $K_\lambda$  on  $\mathbf{P}(\Omega_{X_K/k})$ . Let  $Z_K$  be the closure of the rational points given by the above way. The  $m$ -th power of the fundamental sheaf  $\mathcal{O}_{P_K}(1)$  defines a rational map  $\phi_K$ , which is defined over all the rational points obtained above. Let  $W_K$  denote by the image of  $Z_K$  by  $\phi_K$ . To every  $K$ -rational point  $w$ , Let  $Z_w$  denote the fibre of  $\phi_K$  over  $w$ . One denotes  $Z_K \cap \mathbf{P}(\Omega_{X_K/k})$  by  $\Delta$ .

1. In case  $\dim \phi(Z_K) = \dim Z_K$ , the set of the rational points is bounded. Hence it can be proved.
2. In case  $\text{codim}(Z_K \cap \Delta, Z_K) \geq 2$ , we replace the normalization of  $Z_K$  by  $Z_K, \Delta$ . The exact sequence  $0 \rightarrow f_K^* \mathcal{O}_K \rightarrow \Omega_{X_K/k} \rightarrow \Omega_{X_K/K} \rightarrow 0$  splits over  $Z_w \setminus \Delta_w$ , where  $w \in W_K$ . This splitting extends to  $Z_w$ . The induced map  $Z_K \rightarrow X_K \otimes_K K(w)$  is a projective surjection.

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Hence the exact sequence  $0 \rightarrow f_K^* \mathcal{O}_K \rightarrow \Omega_{X_K/k} \rightarrow \Omega_{X_K/K} \rightarrow 0$  splits over  $X_K$ . Thus Kodaira-Spencer-Grauert's theorem implies that  $X_K \otimes_K K(w)$  is trivial.

3. In case  $\text{codim}(Z_K \cap \Delta, Z_K) = 1$ , it is divided in the following cases.
4. In case  $\dim Z_K > \dim X_K$ , we can find a hyperplane  $H$  on  $Z_K$  such that  $\text{codim}(\Delta \cap H, H) \geq 1$ , on which the induction works.
5. In case  $\dim Z_K = \dim X_K$ , let  $D$  denote  $\pi(\Delta)$ . We may assume that the exact sequence  $0 \rightarrow f_K^* \mathcal{O}_K \rightarrow \Omega_{X_K/k} \rightarrow \Omega_{X_K/K} \rightarrow 0$  splits outside  $D$ .

**Lemma 2.1.** *Let  $A = k[\epsilon]$  and  $\epsilon$  a dual number. Let  $X$  be a smooth projective variety of general type over  $\text{Spec} A$  and  $D$  a divisor on  $X$ . Assume that Kodaira-Spencer class in  $H^1(X_0 \setminus D_0, \Theta_{X_0})$  is zero and that the obstruction of the extension vanishes. Then  $X \setminus D = (X_0 \setminus D_0) \otimes_k A$ . The image of the pluricannonical mapping extends trivially.*

Since the image of the pluricannonical mapping is projective, Kodaira-Spencer and Grauert's theorem implies that the image of the pluricannonical mapping is trivial. Since  $X_K$  is of general type,  $X_K$  is isotrivial.

Therefore  $X/C$  is isotrivial. □

## 2.2 Shafarevitch-type Problem

**Lemma 2.2.** *Let  $k$  be a field of characteristic 0 and all varieties defined over  $k$ . Let  $F : \mathcal{X} \rightarrow C \times D$  be a projective surjective morphism of a projective non singular variety onto a product of two non singular curves  $C$  and  $D$ . Let  $C^\circ$  and  $D^\circ$  be non empty open subset of  $C$  and  $D$ , respectively. Suppose that the restriction  $F^\circ : \mathcal{X}^\circ \rightarrow C^\circ \times D^\circ$  of  $F$  is a projective surjective smooth morphism and that for any point  $t$  of  $C^\circ$   $F_{t*}(\Omega_{\mathcal{X}_t/C})$  is ample, where  $F_t : \mathcal{X}_t \rightarrow C \times \{t\}$ . Assume that  $\Omega_{\mathcal{X}/C \times D}$  is  $F$ -ample. Then  $\mathcal{X}_t/C$  is rigid for  $t$ .*

*Proof.*  $H^1(\mathcal{X}_t, \text{Hom}(\Omega_{\mathcal{X}_t/C}^1, \mathcal{O}_{\mathcal{X}_t})) = H^{d-1}(\mathcal{X}_t, \Omega_{\mathcal{X}_t}^d \otimes \Omega_{\mathcal{X}_t/C}^1) = 0$ , where  $d = \dim \mathcal{X}_t \geq 2$ . □

**Lemma 2.3.** *Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  be projective morphisms between schemes and  $h = g \circ f$ . Let  $\mathcal{E}$  be an  $\mathcal{O}_X$  locally free sheaf of finite rank. Assume that  $\mathcal{E}$  is  $f$ -ample and  $f_* S^n \mathcal{E}$  is locally free and  $g$ -ample for all  $n \geq 1$ . Then  $\mathcal{E}$  is  $h$ -ample.*

*Proof.* Let  $h = g \circ f$ . By hypothesis, for any coherent sheaf  $\mathcal{G}$  the canonical homomorphism

$$g^* g_* \mathcal{G}(S^m f_* S^n \mathcal{E}) \rightarrow \mathcal{G}(S^m f_* S^n \mathcal{E})$$

is surjective for sufficiently large  $m$ , since  $f_* S^n \mathcal{E}$  is  $g$ -ample. Put  $\mathcal{G} = f_* \mathcal{F}(S^k \mathcal{E})$  for a coherent sheaf  $\mathcal{F}$ .  $f^* g^* g_* \mathcal{G}(S^m f_* S^n \mathcal{E}) \rightarrow f^* \mathcal{G}(S^m f_* S^n \mathcal{E})$  is surjective. The composite  $f^* f_* \mathcal{F}(S^k \mathcal{E}) \otimes S^m f_* S^n \mathcal{E} \rightarrow \mathcal{F}(S^k \mathcal{E}) \otimes S^{mn} \mathcal{E} \rightarrow \mathcal{F}(S^{k+mn} \mathcal{E})$  is surjective. Hence

$$f^* g^* g_* \mathcal{G}(S^m f_* S^n \mathcal{E}) \rightarrow f^* g^* g_* f_* \mathcal{F}(S^{k+mn} \mathcal{E}) \rightarrow \mathcal{F}(S^{k+mn} \mathcal{E})$$

is surjective. Thus  $h^* h_* \mathcal{F}(S^{k+mn} \mathcal{E}) \rightarrow \mathcal{F}(S^{k+mn} \mathcal{E})$  is surjective. Therefore  $\mathcal{E}$  is  $h$ -ample. □

**Theorem 2.2.** *Let  $k$  be a field of characteristic 0. Let  $C$  be a projective non singular curve over  $k$ . Let us consider the set of projective smooth surjective morphisms  $f : X \rightarrow C$  satisfying*

1.  $\Omega_{X/C}$  is  $f$ -ample.
2.  $f_* S^m \Omega_{X/C}$  are ample for all  $m \geq 1$ .
3. the self-intersection number  $I(\omega_X)^d$ , where  $d = \dim X$ , is bounded by a number  $B$ .

Then there exist a finite number of families  $\{X/C\}$  satisfying the conditions above.

*Proof.* The set of polynomials in the form  $\chi(X, \omega_X^{\otimes m}) = \dim H^0(X, \omega_X^{\otimes m})$  with  $I(\omega_X^d)$  bounded is finite. Hence the canonical embedding  $\phi : X \rightarrow \mathbf{P}(H^0(X, \omega_X^m))$  can be estimated. Thus one can have a suitable projective space  $P$  which embeds all the  $X$  that form a bounded family. We obtain the theorem by lemmas above and standard arguments.  $\square$

### 2.3 Miscellaneous Remarks

**Theorem 2.3.** *Let  $f : X \rightarrow S$  be a projective connected smooth surjective morphism between non singular varieties with a general fibre of non zero geometric genus. Assume that  $R^d f_* \mathbf{C}$ , where  $d = \dim X/S$ , is a simple sheaf or a constant sheaf. Then  $X/S$  is isotrivial. If  $S$  is simply connected,  $R^d f_* \mathbf{C}$  is constant.*

*Proof.* One sees that  $f_* \omega_{X/S}$  is trivial.  $\square$

**Proposition 2.1.** *Let  $X$  be a projective non singular variety with  $\Omega_X^a$  ample. Let  $S_\lambda$  be the family of non singular subvarieties of dimension  $a$  with  $\dim H^0(S_\lambda, \omega_{S_\lambda}^m)$  a given polynomial in  $m$ . Then the set of  $\{S_\lambda\}$  is bounded.*

*Proof.* There exists a natural surjection  $\Omega_X^a \rightarrow \Omega_{S_\lambda}^a$ . Hence  $\omega_{S_\lambda}$  is ample.  $\chi(S_\lambda, \omega_{S_\lambda}^m) = \dim H^0(S_\lambda, \omega_{S_\lambda}^m)$  is a given polynomial. This completes a proof.  $\square$

**Definition 2.1.** *Let  $D$  be a divisor with only normal crossings. Let  $D_{X/S}$  and  $D_{/S}$  denote the horizontal component and the vertical component of  $D$  with respect to  $f$ , respectively.  $\Omega_{X/S}^a \langle D \rangle = \Omega_{X/S}^a \langle D_{X/S} \rangle ([D_{/S}])$*

**Lemma 2.4.** *Let  $f : X \rightarrow S$  be a proper smooth morphism and  $D$  a divisor with only normal crossings. Let  $\delta : Y \rightarrow X$  be the multi-Kummer covering of  $X$  with respect to the set of components of  $\{D\}$  with Galois group  $G$ . Then  $\delta_* \Omega_{Y/S}^a (\delta^* D)^G = \Omega_{X/S}^a \langle \{D\} \rangle ([D])$ .*

*Proof.* One has the following local formula

$$(\delta_* \Omega_{Y/S}^a (\delta^* D))^G = (\oplus_{i+j=a} \delta_* \Omega_{Y/S}^i \wedge f^* \Omega_S^j (\delta^* D))^G = \\ \oplus_{i+j=a} \Omega_{X/S}^i \langle \{D_{X/S}\} \rangle \wedge f^* \Omega_S^j \langle \{D_{/S}\} \rangle ([D])$$

Put  $j = \dim S$ . One obtains the formula above.  $\square$

**Theorem 2.4.** *Let  $f : X \rightarrow S$  be a projective smooth surjective morphism between non singular quasi-projective varieties over a field of characteristic 0. Then  $f_* S^m \Omega_{X/S}^a$  is weakly positive.*

*Proof.* Let  $\pi : \mathbf{P}(S^m \Omega_{X/S}^a) \rightarrow X$  be the projective bundle over  $X$ . Let  $g$  denote  $\pi \circ f$ . Let  $L$  be an ample invertible sheaf over  $S$ . We can prove it in the category of algebraic stacks. Hence we see that  $L$  is divisible over  $S$ .

Iterating the next lemma  $n$  times, one obtains the weak positivity of

$$f_* S^{m+1} \Omega_{X/S}^a \otimes L^{\otimes (\frac{\alpha m+1}{\alpha(m+1)})^n}$$

Hence choosing  $L$  suitably at the first step, one has the weak positivity of  $f_* S^{m+1} \Omega_{X/S}^a$  itself.  $\square$

**Lemma 2.5.** *If  $f_* S^{m+1} \Omega_{X/S}^a \otimes L$  is weakly positive, then  $f_* S^{m+1} \Omega_{X/S}^a \otimes L^{\otimes \frac{\alpha m+1}{\alpha(m+1)}}$  is weakly positive for any  $\alpha \geq 2$  is weakly positive.*

*Proof.* Consider the natural morphisms

$$\psi_{m,m(m+1)} : \mathbf{P}(S^m \Omega_{X/S}^a) \rightarrow \mathbf{P}(S^{m(m+1)} \Omega_{X/S}^a)$$

and

$$\psi_{m+1,m(m+1)} : \mathbf{P}(S^{m+1} \Omega_{X/S}^a) \rightarrow \mathbf{P}(S^{m(m+1)} \Omega_{X/S}^a).$$

Note that  $\psi_{m,m(m+1)}^* \mathcal{O}_{P_{m(m+1)}}(1) = \mathcal{O}_{P_m}(m+1)$  and  $\psi_{m+1,m(m+1)}^* \mathcal{O}_{P_{m(m+1)}}(1) = \mathcal{O}_{P_{m+1}}(m)$ . Let

$$\mathcal{M} = \mathfrak{I}((\pi_{m(m+1)} \circ f)^*(\pi_{m(m+1)} \circ f)_* \mathcal{O}_{P_{m(m+1)}}(\frac{1}{m}) \rightarrow \mathcal{O}_{P_{m(m+1)}}(\frac{1}{m}))$$

and

$$\mu_{m(m+1)} : P_{m(m+1)} \rightarrow P_{m(m+1)}$$

blow-ups for  $\mu_{m(m+1)}^* \mathcal{M}$  to be an invertible sheaf denoted by  $\mathcal{O}_{P_{m(m+1)}}(\frac{1}{m})$ . Let  $\mu_m : P_m \rightarrow P_{m(m+1)}$  and  $\mu_{m+1} : P_{m+1} \rightarrow P_{m(m+1)}$  be the pullbacks of  $\mu_{m(m+1)}$ ,  $\psi_{m+1,m(m+1)}^* \mathcal{O}_{P_{m(m+1)}}(\frac{1}{m}) = \mathcal{O}_{P_{m+1}}(1)$  and  $\psi_{m,m(m+1)}^* \mathcal{O}_{P_{m(m+1)}}(\frac{1}{m}) = \mathcal{O}_{P_m}(\frac{m+1}{m})$ . By hypothesis,  $S^\beta(S^{\alpha m}(f_* S^{m+1} \Omega_{X/S}^a \otimes L) \otimes L)$  is generically spanned by the global sections over  $S$ . Hence  $\mathcal{O}_{P_{m(m+1)}}(\alpha) \otimes h_{m(m+1)}^* L^{\alpha m+1}$  has a global section on  $P_{m(m+1)}$  and is generically spanned by the global sections on  $h_{m(m+1)}^{-1} S$ . Furthermore we replace  $P_{m(m+1)}$  so that  $\mathcal{O}_{P_{m(m+1)}}(\alpha) \otimes h_{m(m+1)}^* L^{\alpha m+1} = \mathcal{O}_{P_{m(m+1)}}(\alpha(m+1)D)$  and  $[\{D\}]$  forms a normal crossing divisor. Note that  $[D/S] = 0$ . Let  $\delta_{m(m+1)} : Q_{m(m+1)} \rightarrow P_{m(m+1)}$  be the multi-Kummer covering with respect to the set of components of  $\{D\}$ . Let  $\delta_m : Q_m \rightarrow P_m$  and  $\delta_{m+1} : Q_{m+1} \rightarrow P_{m+1}$  be the pullbacks, respectively.

$$(\delta_* \Omega_{Q_m/S}^a (\delta^* (\mathcal{O}_{P_m}(1) \otimes (\delta \circ h_m)^* \frac{\alpha m + 1}{\alpha(m+1)} L - D)))^G =$$

$$\Omega_{P_m/S}^a \langle \{D\} \rangle ([\mathcal{O}_{P_m}(1) \otimes h_m^* (\frac{\alpha m + 1}{\alpha(m+1)} L - D)]) =$$

$$\Omega_{P_m/S}^a \langle \{D_{P_m/S}\} \rangle (\mathcal{O}_{P_m}(1) \otimes h_m^* \frac{\alpha m + 1}{\alpha(m+1)} L - [D_{P_m/S}]) \hookrightarrow \Omega_{P_m/S}^a(1) \otimes h_m^* (\frac{\alpha m + 1}{\alpha(m+1)} L).$$

The last map is generically surjective. By le Potier's theorem, the composite of the following maps is an identity map;

$$f_* \Omega_{X/S}^a \otimes S^m \Omega_{X/S}^a \rightarrow h_{m*} \Omega_{P_m/S}^a(1) \rightarrow g_{m*} \Omega_{P_{m+1}/S}^a(1) = f_* \Omega_{X/S}^a \otimes S^m \Omega_{X/S}^a.$$

Since  $\otimes^{m+1} \Omega_{X/S}^a \rightarrow \Omega_{X/S}^a \otimes S^m \Omega_{X/S}^a \rightarrow S^{m+1} \Omega_{X/S}^a$  has a splitting,  $f_* S^{m+1} \Omega_{X/S}^a \otimes L^{\otimes \frac{\alpha m + 1}{\alpha(m+1)}}$  is weakly positive. This completes the proof of Lemma.  $\square$

**Theorem 2.5.** *For any  $\alpha \geq 2$  there exists a  $\beta > 0$  such that  $f_* S^\beta (S^\alpha \Omega_{X/S}^a) \otimes L^{\otimes \beta}$  is generically spanned by the global sections.*

### 3 Deformation theory of log varieties

Let  $f : X \rightarrow S$  be a proper smooth morphism between smooth varieties and  $A, B$  relative divisors on  $X/S$ . Assume that  $A + B$  is normal crossing. One denotes by  $\Omega_{X/S}^a(A, B)$   $\Omega_{X/S}^a \langle A + B \rangle (-A)$  and by  $\Theta_{X/S}^a(A, B)$   $\text{Hom}(\Omega_{X/S}^a(A, B), \mathcal{O}_X)$ , respectively. For simplicity, one uses  $\Theta_{X/S}(A, B) = \Theta_{X/S}^1(A, B)$  and  $\Omega_{X/S}(A, B) = \Omega_{X/S}^1(A, B)$ . By local coordinates, a local section of  $\Omega_{X/S}^a(A, B)$  (resp.  $\Theta_{X/S}(A, B)$ ) is expressed as

$$u^1(a_{1,2,i_3,\dots,i_a} \frac{du^1}{u^1} \wedge \frac{du^2}{u^2} \wedge du^{i_3} \wedge \dots)$$

(resp.

$$u^2(b_1 \frac{\partial}{\partial u^1} + b_2 \frac{\partial}{u^1 \partial u^2} \dots))$$

, where  $A, B$  is written by  $u^1 = 0, u^2 = 0$ .

**Lemma 3.1.** *Let  $f : X \rightarrow S$  be a proper smooth morphism and  $A, B$  relative divisors on  $X/S$  of dimension  $d = \dim X/S$ . Assume that*

1.  $A_{\text{red}} + B_{\text{red}}$  is normal crossing.

*Then  $\Theta_{X/S}^a(A, B) = \text{Hom}(\Omega_{X/S}^a(A, B), \mathcal{O}_X) = \Omega_{X/S}^{d-a}(B, A)(-K_{X/S}) = \Omega_{X/S}^{d-a}(A, B)(A - B - K_{X/S})$ . The sequence  $0 \rightarrow \Theta_{X/S}(A, B) \rightarrow \Theta_X(A, B) \rightarrow f^*\Theta_S \rightarrow 0$  is exact.*

*Assume furthermore that*

2.  $f_*\mathcal{O}_X = \mathcal{O}_S$

3. Kodaira-Spencer map  $\rho_{X/S}(A, B) : \Theta_S \rightarrow R^1 f_*\Theta_{X/S}(A, B)$  is zero

*Then there exists a lifting of a vector field upon  $f_*\Theta_X(A, B)$ .*

**Lemma 3.2.** *Suppose the same assumption as above. If the Kodaira-Spencer map  $\rho_{X/S}(A, B) : \Theta_S \rightarrow R^1 f_*\Theta_{X/S}(A, B)$  is zero, the pair  $(X, B)$  is locally  $S$ -trivial outside  $A$ .*

**Remark 3.1.** *If, in particular,  $A = 0$ , then the pair  $(X, B)$  is locally  $S$ -trivial.*

*Proof.* In case  $A = 0$ : By Cauchy-Kowarevsky's Theorem,  $X/S$  is locally trivial over  $S$ . Moreover, all integral curves with initial positions in  $B/S$  are stable in  $B/S$ . Hence,  $(X, B)$  is locally  $S$ -trivial.

Locally, one can write down in the following way. Considering  $X/S \times \mathbf{C}$ , one choose regular parameters as unknowns  $u^i$ ,  $u^1 = 0$  associated to the support of  $B$  and a parameter  $t \in \mathbf{C}$  associated to a vector field of  $\Theta_S$ . By Cauchy-Kowarevsky's Theorem, one has a unique local solution in the neighbourhood of the initial condition  $u^i(x, 0) = x^i$ .

$$\frac{du^1}{dt} = (u^1)^{m_1} f^1(t, u)$$

$$\frac{du^i}{dt} = f^i(t, u)$$

If  $x^1 = 0$ ,  $(\frac{d}{dt})^n u^1 = 0$  at  $u^1(x, 0) = 0$ . Hence  $e^{h\frac{d}{dt}} u^1 = u^1(x, h) = 0$  for a constant  $h$ . Therefore  $(X, B)$  is locally  $S$ -trivial.

In case  $B = 0$ : If the velocity is changed together with space, the relation between time and space breaks but the integral curves in space are kept invariant. All integral curves with initial positions in  $A/S$  have time interval of zero; hence  $A/S$  can be deformed.

Locally, one can write down in the following way. Considering  $X/S \times \mathbf{C}$ , one choose regular parameters as unknowns  $u^i$ ,  $u^1 = 0$  associated to the support of  $A$  and a parameter  $t \in \mathbf{C}$  associated to a vector field of  $\Theta_S$ . By assumption, there exists a projectable vector field with poles along  $u^1 = 0$ . The differential system is the following:

$$\frac{du^1}{dt} = f^1(t, u)$$

$$\frac{du^i}{dt} = \frac{f^i(t, u)}{(u^1)^{m_1}}$$

The system of  $B = 0$  is obtained by multiplying  $\frac{1}{(u^1)^{m_1}}$  to the system of  $A = 0$  in the right side. No integral curve can be throughout of  $A$ . Hence one has a unique local solution in the neighbourhood of the initial condition  $u^i(0, x) = x^i$  if  $x^1 \neq 0$ .

In the neighbourhood of the initial condition  $u^1(x, 0) = x^1 = 0$ , one can transform this differential system into the following form.

$$\frac{du^1}{(u^1)^{m_1} f^1(t, u)} = \frac{du^i}{f^i(t, u)} = \frac{dt}{(u^1)^{m_1}}$$

The trajectory is in  $t = 0$  and  $u^1 = 0$ . Thus  $X \setminus A$  is  $S$ -trivial.  $A/S$  has no restriction. Therefore they determine a birational mapping among  $X_t$  ( $|t| < \epsilon$ ) for sufficiently small  $\epsilon > 0$ .

In general:  $(X, B)/S$  is  $S$ -trivial outside  $A$ .

□

Let  $f : X \rightarrow S$  be a projective smooth connected surjective morphism between quasi-projective non singular varieties over the field of the complex numbers. Let  $A, B$  be relative divisors on  $X/S$ . Assume  $A_{red} + B_{red}$  is normal crossing. The differential of the period map of the mixed Hodge structure  $R^a f_* \Omega_{X/S}^i(A, B)$  gives a homomorphism

$$\Theta_S \rightarrow \bigoplus_{i+j=a} \text{Hom}_{\mathcal{O}_X}(R^j f_* \Omega_{X/S}^i(A, B), R^{j+1} f_* \Omega_{X/S}^{i-1}(A, B))$$

The variation of the bottom Hodge filtration  $f_* \Omega_{X/S}^a(A, B)$  is known by the homomorphism map  $\Theta_S \rightarrow \text{Hom}(f_* \Omega_{X/S}^a(A, B), R^1 f_* \Omega_{X/S}^{a-1}(A, B))$ .

**Lemma 3.3.** *Let  $f : X \rightarrow S$  be a projective smooth connected surjective morphism between non singular quasi-projective varieties over the field of the complex numbers. Let  $A, B$  be relative divisors on  $X$ . Assume*

1.  $A_{red} + B_{red}$  is normal crossing
2.  $f_* \Omega_{X/S}^a = \mathcal{O}_{X/S}$  as locally constant sheaves.

Then the map

$$\Theta_S \rightarrow \text{Hom}(f_* \Omega_{X/S}^a(A, B), R^1 f_* \Omega_{X/S}^{a-1}(A, B)) = R^1 f_* \Theta^{d-a+1}(A, B)(K_{X/S} - A + B)$$

is zero.

**Lemma 3.4.** *Let  $f : X \rightarrow S$  be a projective smooth connected surjective morphism between non singular quasi-projective varieties over the field of the complex numbers. Let  $D$  be a relative divisor on  $X/S$ . Assume*

1.  $K_{X/S} = D$
2.  $D_{red}$  is normal crossing
3.  $f_* \omega_{X/S} = \mathcal{O}_X$  as locally constant sheaves.

Then  $X/S$  is isotrivial.

*Proof.* By assumption, the variation of the bottom Hodge filtration does not move. Hence the homomorphism  $\Theta_S \rightarrow \text{Hom}(f_* \Omega_{X/S}^d(D, 0), R^1 f_* \Omega_{X/S}^{d-1}(D, 0))$  is zero, where  $d = \dim X/S$ . Thus the homomorphism

$$\Theta_S \rightarrow \text{Hom}(f_* \Omega_{X/S}^d(D, 0), R^1 f_* \Theta_{X/S}(D, 0)(-D + K_{X/S}))$$

is zero. Note that  $\Omega_{X/S}^d(D, 0) = \omega_{X/S}$ . Therefore  $X/S$  is isotrivial.  $\square$

## 4 Application

**Theorem 4.1.** *Let  $f : X \rightarrow S$  be a projective connected surjective morphism between non singular quasi-projective varieties over the field of the complex numbers. Assume that*

1. a general fibre  $X_s$  is of Kodaira dimension 0
2.  $f_* \omega_{X/S}^{\otimes m} = \mathcal{O}_S$  for some  $m > 0$  as locally constant sheaves, hence for all  $m > 0$ .

Then  $X/S$  is isotrivial.

*Proof.* Step1. If necessary, it is possible to replace  $X$  and  $S$  by a proper birational variety and a suitable open subvariety, respectively. Thus one may assume that

1.  $f$  is smooth

2. there exists an  $m$  such that for any  $k > 0$   $\omega_{X/S}^{km} = \mathcal{O}_X(\sum_j k\nu_j D_j)$ , where the  $D_j$  is a relative normal crossing divisor and the right hand term expresses the stable fixed component

By taking the multi-Kummer covering  $\nu : Y = \text{Spec } \mathcal{O}_X[T_j]/(T_j^{\mu_j} - D_j) \rightarrow X$  such that  $\mu_j\nu_j + \mu_j - 1 = n$  for all  $j$ , one has  $K_{Y/S} = nE$ , where  $E = \sum_j D'_j$  is a relative normal crossing divisor and  $\mu_j D'_j = D_j|_Y$ . Note that  $n$  can be chosen to be an odd number.

Step2. One sees that  $(\nu_*(\omega_{Y/S}((n-1)K_{Y/S} - \frac{n-1}{n}E)))^G = \omega_{X/S}^{\otimes n}$ . Applying the lemma above, one obtains isotriviality of  $Z/S$ . Thus  $X/S$  is isotrivial.  $\square$

Special case of Viehweg Conjecture with a general fibre of  $\kappa(\omega_{X_s}) = 0$  implies Viehweg Conjecture over curves. General case reduces to curves case.

**Theorem 4.2.** *Let  $C$  be a projective non singular curve. Let  $f : X \rightarrow C$  be a fibre space with a general fibre of Kodaira dimension non negative. Then there exists an  $m > 0$  such that  $\kappa(\det f_*\omega_{X/C}^{\otimes m}) \geq \text{var}(X/C)$ .*

*Proof.* It suffices to show that  $\text{var} X/C = 0$  when  $f_*\omega_{X/C}^{\otimes m}$  is locally constant. Let  $h : Z \rightarrow C$  be the image of a rational map defined by  $f^*f_*\omega_{X/C}^{\otimes m} \rightarrow \omega_{X/C}^{\otimes m}$ . Replacing  $X$  by a resolution of indeterminacy of the rational map one denotes the morphism  $X/Z$  by  $g$ . Then  $Z$  is trivial. Put  $Z = C \times W$ . Let  $h_X : X \times W \rightarrow X$  be the pullback of  $h : Z \rightarrow C$  and  $f_X : X \times W \rightarrow X$  the pullback of  $f$ . For any point  $w$  of  $W$ , one has the commutative triangle of the pullbacks among  $h_w : X \times \{w\} \rightarrow X_w$ ,  $g_w : X_w \rightarrow C \times \{w\}$  and  $f_w : X \times \{w\} \rightarrow C \times \{w\}$ . When  $\dim Z/C < \dim X/C$ , one has  $g_w*\omega_{X_w/C}^{\otimes m} \hookrightarrow f_w*\omega_{X_w/C}^{\otimes m}$ . If  $g_w*\omega_{X_w/C}^{\otimes m}$  contains a big invertible sheaf, then  $f_w*\omega_{X_w/C}^{\otimes m}$  contains it, which is a contradiction. By induction of Viehweg Conjecture about relative dimension,  $\kappa(\det g_*\omega_{X_w/C}^{\otimes m}) = 0$  implies  $\text{var}(X_w/C) = 0$ . Hence  $\text{var}(X/C) = 0$ . Thus it remains to prove the case  $\dim W = 0$ . It is the case of Kodaira dimension zero of a general fibre. It was already proved.  $\square$

**Theorem 4.3.** *Let  $f : X \rightarrow C$  be a fibre space with a general fibre of Kodaira dimension non negative. Then for any  $m > 0$   $\det f_*\omega_{X/S}^{\otimes m}$  is big unless it is zero.*

**Theorem 4.4.** *Let  $f : X \rightarrow S$  be a proper connected surjective morphism between non singular varieties. Let  $H$  be a non singular ample divisor on  $X$ . Let  $L = \mathcal{O}_X(m(K_{X/S} + H))$  for  $m > 0$  and  $L(-i) = L(-iH)$  for  $i > 0$ . Then  $R^k f_* L(-i)$  is torsion free for all  $k \geq 0$ .*

*Proof.* Note that by Kollar's torsion freeness theorem,  $R^k f_* \mathcal{O}_X(m(K_{X/S} + H))$  is torsion free. There exists a long exact sequence

$$\begin{aligned} \cdots \rightarrow R^k f_* L(-i-1) \rightarrow R^k f_* L(-i) \rightarrow R^k f_* L(-i)_H \rightarrow R^{k+1} f_* L(-i-1) \rightarrow \\ R^{k+1} f_* L(-i) \rightarrow \cdots \end{aligned}$$

Let  $M_S$  be the rational function field. Tensoring this  $M_S$  with the sequence above, one obtains a corresponding long exact sequence. One proceeds by induction about  $k, i$  and dimension of  $X$ . Assume  $R^k f_* L(-i)$ ,  $R^k f_* L(-i)_H$  and  $R^{k+1} f_* L(-i)$  are torsion free. One shall show that  $R^{k+1} f_* L(-i-1)$  is also torsion free. Let  $\psi$  be a local section of  $R^{k+1} f_* L(-i-1)$  whose canonical image in  $R^{k+1} f_* L(-i-1) \otimes M_S$  is zero. There exists a local section  $\phi$  in  $R^k f_* L(-i)_H$  which maps to  $\psi$ . Then there exists a local section  $a(t)$  of  $\mathcal{O}_S$  such that  $a(t)\phi$  is the image of a certain local section  $\Psi$  of  $R^k f_* L(-i)$ . By transition functions  $g_{\mu\nu}$ ,  $\Psi$  can be written in the form  $\Psi_\mu = g_{\mu\nu}\Psi_\nu$  for the family  $U_\mu$  of the neighbourhoods of the pullback of an affine open subset of  $S$ . Further,  $\Psi_\mu = a(t)\Phi_\mu + \xi_\mu$ , where  $\xi_\mu$  is a local section of the image of  $R^k f_* L(-i-1)$  on  $U_\mu$ . Take a point  $t_0$  on  $S$  such that  $a(t_0) = 0$ . Hence one has  $\xi_\mu = g_{\mu\nu}\xi_\nu$  on the fibre of  $t_0$ . Thus there exists a local section  $\xi_\mu = g_{\mu\nu}\xi_\nu$  on an affine open of  $t_0$ . Therefore  $a(t)\phi_\mu = g_{\mu\nu}(a(t)\phi_\nu)$  determines a local section  $R^k f_* L(-i)$  on an affine open subset of  $t_0$ . The torsion freeness of  $R^k f_* L(-i)$  implies that  $\Phi_\mu = g_{\mu\nu}\Phi_\nu$ . The image of this local section  $\Phi_\mu = g_{\mu\nu}\Phi_\nu$  coincides with the given  $\phi$  since  $R^k f_* L(-i)_H$  is torsion free. Hence  $\psi = 0$ . This completes the proof.  $\square$

**Lemma 4.1.** *Let  $f : X \rightarrow S$  be a fibre space and  $A$  a general hyperplane of  $S$ . Then*

$$0 \rightarrow R^i f_* \omega_{X/S}^{\otimes m}(-A) \rightarrow R^i f_* \omega_{X/S}^{\otimes m} \rightarrow R^i f_{A*} \omega_{X_A/A}^{\otimes m} \rightarrow 0$$

*In particular, one obtains*

$$0 \rightarrow f_* \omega_{X/S}^{\otimes m}(-A) \rightarrow f_* \omega_{X/S}^{\otimes m} \rightarrow f_{A*} \omega_{X_A/A}^{\otimes m} \rightarrow 0$$

*Proof.* By the long exact sequence

$$\begin{aligned} \cdots \rightarrow R^i f_* \omega_{X/S}^{\otimes m}(-A) \rightarrow R^i f_* \omega_{X/S}^{\otimes m} \rightarrow R^i f_{A*} \omega_{X_A/A}^{\otimes m} \rightarrow R^{i+1} f_* \omega_{X/S}^{\otimes m}(-A) \rightarrow \\ R^{i+1} f_* \omega_{X/S}^{\otimes m} \rightarrow \cdots \end{aligned}$$

and the torsion freeness of  $R^i f_* \omega_{X/S}^{\otimes m}$ , one can conclude the proof.  $\square$

**Lemma 4.2.** *The following formulas hold*

$$f_* \omega_{X/S}^{\otimes m}|_A = f_{A*} \omega_{X_A/A}^{\otimes m}$$

$$\det f_* \omega_{X/S}^{\otimes m}|_A = \det f_{A*} \omega_{X_A/A}^{\otimes m}$$

**Lemma 4.3.** *There exists a general curve  $C$  on  $S$  such that*

$$H^0(S, (\det f_* \omega_{X/S}^{\otimes m})^{\otimes \ell}) \rightarrow H^0(C, (\det f_{C*} \omega_{X_C/C}^{\otimes m})^{\otimes \ell})$$

*is surjective.*

*Proof.* By the exact sequence  $0 \rightarrow \det f_* \omega_{X/S}^{\otimes m}(-A) \rightarrow \det f_* \omega_{X/S}^{\otimes m} \rightarrow \det f_{A*} \omega_{X_A/A}^{\otimes m} \rightarrow 0$ , one obtains

$$\begin{aligned} 0 \rightarrow H^0(S, \det f_* \omega_{X/S}^{\otimes m}(-A)) \rightarrow H^0(S, \det f_* \omega_{X/S}^{\otimes m}) \rightarrow \\ H^0(A, \det f_{A*} \omega_{X_A/A}^{\otimes m}) \rightarrow H^1(S, \det f_* \omega_{X/S}^{\otimes m}(-A)) \rightarrow \cdots \end{aligned}$$

One can choose  $A$  such that

$$H^1(\det f_* \omega_{X/S}^{\otimes m}(-A)) = H^{d_S-1}(\det f_* \omega_{X/S}^{\otimes m})^{\otimes -\ell}(A) = 0$$

if  $\dim S \geq 2$ .  $\square$

**Lemma 4.4.** *If  $\text{var } X/S = \dim S$ , then  $\kappa(\det f_* \omega_{X/S}^{\otimes m}) = \dim S$  if  $f_* \omega_{X/S}^{\otimes m} \neq 0$ .*

*Proof.* Assume that  $\kappa(\det f_* \omega_{X/S}^{\otimes m}) < \dim S$ . Consider the rational map determined by  $\det f_* \omega_{X/S}^{\otimes m}$ . If necessary, change  $S$  by blow-ups.  $X/S$  does not move birationally along a general fibre of the rational map, which contradicts the hypothesis  $\text{var}(X/S) = \dim S$ .  $\square$

## 5 Minimal Models and Logarithmic structures

An open embedding  $U \hookrightarrow X$  is defined to be a toroidal embedding if locally in the étale topology it is isomorphic to a torus embedding  $T \hookrightarrow V$ . Toroidal embedding without self-intersection is called strict toroidal embedding. We denote  $X \setminus U$  by  $D$ . We will sometimes denote the toroidal embedding by  $(X, D)$ .

**Theorem 5.1 (KKMS).** *To every strict toroidal embedding  $U \subset X$ , we can associate a conical polyhedral complex with integral structure  $\Delta = (|\Delta|, \sigma^Y, M^Y)$  whose cells are 1-1 correspondence with the strata of  $X$ .*

Abramovich and De Jong proved the following result.



1. By semistable reduction one obtains a surjective morphism  $X' \rightarrow P'$  of relative dimension 1 after a base change  $P' \rightarrow P$  with Galois group  $G$ .
2. By induction one assumes that  $P$  is smooth and that the discriminant locus  $X' \rightarrow P$  is a strict normal crossing divisor.
3. By blow-ups  $X'/G$  turns to be toroidal.

By the famous Reid's argument to the proof of the flip conjectures in the case of toric varieties one can conclude the flip conjectures generally. The contraction of an extremal ray is toroidal and gives a toroidal image. The existence of a flip comes from one of the two distinct simplicial subdivisions. The termination of flips is in the same way. The existence of minimal nef model is shown easily as log algebraic stacks but as varieties by means of unique local analytic continuation of patching of schemes of affine monoid rings.

**Remark 5.1.** *Flip conjectures are in the affirmative.*

**Definition 5.1.** 1. Let  $P, Q$  be finitely generated integral monoids,  $Q \rightarrow P$  a homomorphism and  $\mathcal{O}_S$  a commutative ring. The morphism of natural log schemes  $(\text{Spec } \mathcal{O}_S[P], P) \rightarrow (\text{Spec } \mathcal{O}_S[Q], Q)$  is said to be log flat (resp. log ramified) if the kernel (resp. the cokernel) of  $Q^{gr} \rightarrow P^{gr}$  is finite group whose order is invertible in  $\mathcal{O}_S$ .

2. Let  $f : (X, M) \rightarrow (Y, N)$  be a morphism of fine log  $S$ -schemes and  $x$  a point of  $X$ .  $f$  is defined to be log flat (resp. log ramified) if  $(X, M) \rightarrow (Y, N)$  is étale locally log flat (resp. log ramified) and if  $X \rightarrow Y \times_{\text{Spec } \mathcal{O}_S[N]} \text{Spec } \mathcal{O}_S[M]$  is flat (resp. ramified).

We will call an algebraic space a scheme.

**Definition 5.2.** An  $S$ -stack  $(\mathcal{X}, \mathcal{M})$  in the category of Kato's log schemes is called log algebraic when satisfying the following two conditions

1. the 1-morphism of  $S$ -stacks of diagonal

$$(\mathcal{X}, \mathcal{M}) \rightarrow (\mathcal{X}, \mathcal{M}) \times_S (\mathcal{X}, \mathcal{M})$$

is representable, separated and quasi-compact,

2. there exists an  $S$ -log scheme  $(X, M)$  and a 1-morphism of  $S$ -stacks

$$(X, M) \rightarrow (\mathcal{X}, \mathcal{M})$$

which is representable, surjective and log smooth.

**Lemma 5.1.** One can replace a log smooth morphism by that of log flat and locally of finite presentation.

**Remark 5.2.** If necessary, we replace étale topology by f.p.p.f topology.

**Example 5.1.** Let  $k$  be a field and  $X$  a fine log scheme over  $k$  locally of finite type.  $(X, M)$  is log flat if and only if f.p.p.f locally on  $X$ , there exists a finitely generated integral monoid  $P$  and a flat morphism  $X \rightarrow \text{Spec } k[P]$  with  $M = P\mathcal{O}_X^* \subset \mathcal{O}_X$ . Thus  $(X, M)$  corresponds to a generalized toroidal embedding which is locally given by the open immersion

$$X \times_{\text{Spec } k[P]} \text{Spec } k[P^{gr}] \subset X.$$

**Theorem 5.2.** Let  $k$  be an algebraically closed field,  $X$  a complete variety over  $k$  and  $Z$  a closed subset of  $X$ , distinct of  $X$  itself. Then there exists a modification  $\phi : X' \rightarrow X$  such that

1.  $X'$  is smooth,

2.  $\phi^{-1}(Z)$  is the support of a divisor with normal crossings.

*Proof.* We give an outline of the proof. By de Jong, there exists an alteration of  $X$  satisfying the condition of the theorem. To this alteration corresponds a log smooth variety. By the Galois group of the Galois closure of the alteration over the rational function field of  $X$ , one makes a log variety as quotients. One can associate a log flat variety  $(X', M)$  to this log variety by changing monoids. Etale locally on  $X'$ ,  $\mathcal{O}_{X'} \otimes k(y)$  is separable over  $k(y)$  for a closed point  $y$  of  $\text{Spec } k[P]$  since  $k$  is algebraically closed. To be geometrically regular is stable by generization. Hence this log flat variety turns out to be log smooth. Hence the forgetful variety  $X'$  becomes smooth after modification by [KKMS].  $\square$

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