ALGEBRAIC CHAMPS AND KUMMER COVERINGS

Kazuhisa MAEHARA

ABSTRACT. In this article, the author studies the conditions such that special stacks are algebraic and proves that Kummer coverings form algebraic stacks([La2]). The notion of K.Kato's logarithmic spaces([Ka]) is enlarged to work in the category of algebraic stacks. By virtue of these notions, he constructs an endomorphism assumed in Kähler analogue of certain conjectures of Weil([Ser]) by Serre.

1. INTRODUCTION

A French word "champs"([La2]) means stacks in English but we prefer champs to stacks because champs have original flavour. Kummer coverings are often recognized to play a rôle in algebraic geometry as curvature does so in differential geometry. Kummer coverings are however not well defined globally even though they are able to be defined well locally. Grothendieck and Murre showed that Kummer coverings form gerbes ([Gir]), ([GM]) and we show that they are algebraic in this article. Intuitively, champs can be reduced to many gerbes([Gir]) by "Stein factorization." The notion of algebraic champs has not yet widely known so we refer to Laumon's work ([La2]). Varieties with logarithmic connections and Kummer coverings seem to be curvature and tensor calculations. K.Kato and others ([Ka]) generalize varieties with logarithmic connections widely into varieties with logarithmic structure or Kato's logarithmic spaces. As one of main theorem in mixed Hodge theory, we state the degeneration of Hodge spectral sequence of Kato's logarithmic spaces disguised into algebraic champs. Hodge theory is not purely algebraically constructed in the category of complex algebraic geometry. We want to make it purely algebraic even in characteristic zero. Toward algebraic proof, upon algebraic champs we construct endomorphisms which Serre assumed in the paper treating Kähler analogue of certain conjectures of Weil ([Ser]).

2. ALGEBRAIC CHAMPS

Let S be a base scheme. We denote by $(\mathrm{Aff})/S$ the category of affine schemes with a morphism of schemes over S. We endow (Aff/S) with the fppf topology.

Definition 2.1 ([Gir], [La2]). The 2-category of S-groupoids

Let $\mathcal{X}$ be a category and α a functor from $\mathcal{X}$ to (Aff/S). The functor $\alpha : \mathcal{X} \to (\mathrm{Aff}/S)$ is said to be an S-groupoid if the following holds:

(1) for any morphism $\phi : V \to U$ and any object $x \in \mathrm{ob}\mathcal{X}$ with $\alpha(x) = U$, there exist an object y and a morphism $f : y \to x$ in $\mathcal{X}$;

(2) for any missing triangle $f : y \to x, h : z \to x$ in $\mathcal{X}$ and for any morphism ψ with $\alpha(f) \circ \psi = \alpha(h)$ in (Aff/S), there exists a unique morphism $g : z \to y$ in $\mathcal{X}$ such that $h = f \circ g$ and that $\alpha(g) = \psi$.

* Tokyo Institute of Polytechnics
Received Oct. 3, 1995

Definition 2.2 ([La2], [Gir]**(for the groupoid to be a champ)).** An S-groupoid $\alpha : \mathcal{X} \to (\mathrm{Aff}/S)$ is said to be an S-prechamp if for any $U \in \mathrm{ob}(\mathrm{Aff}/S)$ and $x, y \in \mathrm{ob}\mathcal{X}$

$$\underline{Isom}(x,y) : (\mathrm{Aff}/S) \to (Set)$$
$$(V \to U) \mapsto \mathrm{Hom}_{\mathcal{X}_V}(x_V, y_V)$$

becomes a sheaf. An S-prechamp is called an S-champ if for any covering family

$$(V_i \xrightarrow{\phi_i} U) \in (\mathrm{Aff}/S)$$

every descent datum (x_i, f_{ji}) is effective. That is, given descent data (x_i, f_{ji}) such that

$$f_{ji} : (x_i|V_{ji}) \simeq (x_j|V_{ji})$$

with cocycle conditions

$$(f_{ki}|V_{kji}) = (f_{kj}|V_{kji}) \circ (f_{ji}|V_{kji})$$

where

$$V_{ji} = V_j \times_U V_i, \qquad V_{kji} = V_k \times_U V_j \times_U V_i$$

there exist $x \in \mathrm{ob}\mathcal{X}_U$ and isomorphisms

$$f_i : x|V_i \simeq x_i$$

in $\mathcal{X}_{V_i}$ such that

$$(f_j|V_{ji}) = f_{ji} \circ (f_i|V_{ji})$$

We refer to the definition of champs for fibered categories([Gir],[La2]).

Definition 2.3 (Descent([Gir]**)).** Let $\mathcal{E}$ be a category and $\mathcal{F}$ an $\mathcal{E}$-fibered category. Let $U \in \mathrm{ob}\ \mathcal{E}$, R a crible of $\mathcal{E}/U$. R is said to be a crible of $\mathcal{F}$-i-descent if the restriction functor

$$\mathrm{Cart}_{\mathcal{E}}(\mathcal{E}/U, \mathcal{F}) \to \mathrm{Cart}_{\mathcal{E}}(R, \mathcal{F})$$

is i-faithful for $i = 0, 1, 2$ which means "faithful, fully faithful, an equivalence", respectively.

Definition 2.4 ($\mathcal{E}$-(pre)Champs). An $\mathcal{E}$-fibered category $\mathcal{F}$ is said to be an $\mathcal{E}$-champ(resp. $\mathcal{E}$-prechamp) if for any object $U \in \mathrm{ob}(\mathcal{E})$ and any raffinement of U, the restriction functor

$$\mathrm{Cart}_{\mathcal{E}}(\mathcal{E}/U, \mathcal{F}) \to \mathrm{Cart}_{\mathcal{E}}(R, \mathcal{F})$$

is an equivalence(resp. fully faithful).

Theorem 2.1 ([Gir]). Given any $\mathcal{E}$-prechamp $\mathcal{F}$ there exists the canonically associated $\mathcal{E}$-champ $\mathcal{F}^a$ with $i : \mathcal{F}^a \to \mathcal{F}$ such that for any $\mathcal{E}$-champ $\mathcal{G}$ the following holds;

$$Hom_{\mathcal{E}}(\mathcal{F}^a, \mathcal{G}) \to Hom_{\mathcal{E}}(\mathcal{F}, \mathcal{G})$$

Definition 2.5 ([La2]). An S-champ $\mathcal{X}$ is said to be representable ([La2]) if there exist an algebraic space X over S and a 1-isomorphism $X \simeq \mathcal{X}$ of S-champs. A 1-morphism of S-champs $F : \mathcal{X} \to \mathcal{Y}$ is said to be representable if for any $U \in \mathrm{ob}(\mathrm{Aff}/S)$ and any $y \in \mathrm{ob}\mathcal{Y}_U(i.e., y : U \to \mathcal{Y}$, the fibre product

$$U \times_{y, \mathcal{Y}, F} \mathcal{X}$$

is representable.

3. The 2-category of algebraic S-champs

Definition 3.1 ([Gir]). A (quasi-separated) algebraic S-champ is an S-champ $\mathcal{X}$ satisfying the following

(1) the diagonal 1-morphism of S-champs
$$\mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$$
is representable, separated and quasi-compact

(2) there exists an algebraic space X over S and a 1-morphism
$$P : X \to \mathcal{X}$$
which is representable, surjective and smooth.

Definition 3.2 ([La2]). An algebraic S-space X is said to have the property P if for a presentation $X \xrightarrow{F} \mathcal{X}$, the algebraic S-space X has the property P.

Theorem 3.1 (Artin's criterion). In the situation above in place of (2) if there exists an algebraic space Y over S and a 1-morphism of S-champs
$$Q : Y \to \mathcal{X}$$
which is representable, faithfully flat and locally of finite presentation, then $\mathcal{X}$ is an algebraic champ over S.

4. Quasi-coherent sheaves over an algebraic S-champ

Definition 4.1 (Definition-Proposition). Let $\mathcal{X}$ be an algebraic S-champ. We call it a big smooth site and denote it by $Lis/\mathcal{X}$ the site defined as follows. The objects of the category of $Lis/\mathcal{X}$ are the smooth 1-morphisms $u : \mathcal{U} \to \mathcal{X}$ denoted by $(\mathcal{U}, u)$ and a morphism of $(\mathcal{U}, u)$ into $(\mathcal{V}, v)$ in $\mathcal{X}$ is a pair (ϕ, α) consisting of a smooth 1-morphism of algebraic S-champs $\phi : \mathcal{U} \to \mathcal{V}$ and a 2-isomorphism $\alpha : u \mapsto v \circ \phi$. The topology over $\mathcal{X}$ is generated by the covering family $Cov(\mathcal{U}, u) = \{(\phi_i, \alpha_i) : ((\mathcal{U}_i, u_i) \to (\mathcal{U}, u))\}$ such that the 1-morphism of algebraic S-champs $\sqcup\phi_i : \mathcal{U}_i \to \mathcal{U}$. We denote by $\mathcal{X}_{lis}$ the full subcategory of $Lis/\mathcal{X}$ whose objects are $\{(U, u)\}$ where $U \in \mathrm{ob}(\mathrm{Aff}/S)$. Then the functor of inclusion $\mathcal{X}_{lis} \to Lis/\mathcal{X}$ induces the equivalence of their topoi.

One has the structure sheaf of rings $\mathcal{O}_{\mathcal{X}}$ over $\mathcal{X}_{lis}$ defined by
$$\Gamma((U, u), \mathcal{O}_{\mathcal{X}}) = \Gamma(U, \mathcal{O}_U)$$
for every $(U, u) \in \mathrm{ob}\mathcal{X}_{lis}$ and denote by $Mod(\mathcal{O}_{\mathcal{X}_{lis}})$ the abelian category of $\mathcal{O}_{\mathcal{X}_{lis}}$-Modules and by $D(\mathcal{O}_{\mathcal{X}_{lis}})$ its derived category.

5. Local construction

Consider the fibered category over (Aff/S)
$$\underline{\mathrm{Ch}}$$
defined as follows. For every $U \in \mathrm{ob}(\mathrm{Aff}/S)$, $\underline{\mathrm{Ch}}_U$ is the category of U-champs. For each morphism $\phi : V \to U$ in (Aff/S), the functor ϕ^* is the functor $V \times_{\phi, U} (-)$.

Definition 5.1 ([La2]). A local construction over S-champ $\mathcal{X}$ is a cartesian functor

$$\underline{\mathcal{Y}} : \mathcal{X} \to \underline{\mathrm{Ch}}$$

Precisely, a local construction $\underline{\mathcal{Y}}$ over $\mathcal{X}$ is given by a functor

$$\underline{\mathcal{Y}}_U : \mathcal{X}_U \to \underline{\mathrm{Ch}}_U$$

and for every morphism $\phi : V \to U$ in (Aff/S) by an isomorphism of functors ϵ_ϕ

$$\mathcal{X}_U \xrightarrow{\phi^*} \mathcal{X}_V \xrightarrow{\underline{\mathcal{Y}}_U} \underline{\mathrm{Ch}}_V$$

onto

$$\mathcal{X}_U \xrightarrow{\underline{\mathcal{Y}}_U} \underline{\mathrm{Ch}}_U \xrightarrow{V \times_{\phi, U} (-)} \underline{\mathrm{Ch}}_V$$

such that for each morphism $\psi : W \to V$ in (Aff/S)

$$\epsilon_{\phi \circ \psi} = (W \times_{\psi, V} \epsilon_\phi) \circ \psi(\phi^*).$$

We can associate to every local construction $\underline{\mathcal{Y}}$ over $\mathcal{X}$ the S-fibered category over $\mathcal{X}$

$$F : \mathcal{Y} \to \mathcal{X}.$$

For every $U \in ob(\mathrm{Aff}/S)$, the fibered category $\mathcal{Y}_U$ is made of the pairs (x, y) where $x \in ob\mathcal{X}_U$ and $y \in \mathrm{Cart}_{(\mathrm{Aff})/U}((\mathrm{Aff})/U, \underline{\mathcal{Y}}_U(x))$. For every morphism $\phi : V \to U$ in (Aff/S) and every $(x, y) \in \mathrm{ob}\mathcal{Y}_U$, $\phi^*(x, y) = (\phi^* x, y')$ where y' is the cartesian section of $\underline{\mathcal{Y}}_V(\phi^* x)$ over V induced by y through the isomorphism

$$\epsilon(x) : \underline{\mathcal{Y}}_V(\phi^* x) \xrightarrow{\sim} V \times_{\phi, U} \underline{\mathcal{Y}}_U(x).$$

F is defined by $F(x, y) = x$ for every $(x, y) \in \mathrm{ob}\mathcal{Y}_U$ and every $U \in \mathrm{ob}(\mathrm{Aff}/S)$.

Proposition 5.1. The S-fibered category $\mathcal{Y}$ above is an S-champ.

Let $\mathcal{X} \to S$ be an algebraic S-champ and $\mathcal{A}$ a champ of coherent $\mathcal{O}_X$-Algebras. A representation $\underline{\mathcal{A}}$ of $\mathcal{X}$ is given by

$$\underline{\mathcal{A}}_U(x) = \mathrm{Cart}_{U_{lis}}(U_{lis}, x^* \mathcal{A})$$

where $x \in \mathrm{ob}\mathcal{X}_U, U \in (\mathrm{Aff}/S)$.

Definition 5.2.

$$\underline{\mathcal{Y}}_U(x) = \mathrm{Spec}(\underline{\mathcal{A}}_U(x))$$

Proposition 5.2 ([La2],[Gir]). Let $\mathcal{X}$ be an S-champ. The following conditions are equivalent:

(1) the 1-morphism of S-champs

$$\mathcal{X} \xrightarrow{\Delta} \mathcal{X} \times_S \mathcal{X}$$

is representable

(2) for every $U \in ob(\mathrm{Aff}/S)$ and all $x, y \in ob\mathcal{X}_U$, the sheaf $\underline{\mathrm{isom}}(x, y)$ over (Aff/S) is representable by an algebraic U-space;

(3) for every $U \in ob(\mathrm{Aff}/S)$ and every $x \in ob\mathcal{X}_U$, the 1-morphism of S-champs $U \xrightarrow{x} \mathcal{X}$ is representable;

(4) for every algebraic U-space, every 1-morphism of S-champs $P : X \to \mathcal{X}$ is representable.

Definition 5.3. An S-champ $\mathcal{X}$ is said to be representable if there exists an algebraic S-space X and a 1-isomorphism $X \cong \mathcal{X}$ of S-champs.

A 1-morphism of S-champs $\mathcal{X} \xrightarrow{F} \mathcal{Y}$ is said to be representable if for every $U \in ob(\mathrm{Aff}/S)$, and every $y \in ob\mathcal{Y}_U$, considered as 1-morphism of S-champs from U into $\mathcal{Y}$, the fibre product $U \times_{y,\mathcal{Y},F} \mathcal{X}$ is representable.

Precisely saying, an algebraic S-champ $\mathcal{X}$ is representable if for every $U \in ob(\mathrm{Aff}/S)$, the category $\mathcal{X}_U$ is equivalent to a discrete category X_U and if the functor

$$(\mathrm{Aff}/S) \longrightarrow (set)$$

$$U \mapsto obX_U$$

is represented by an algebraic S-space X.

A 1-morphism of S-champs $\mathcal{X} \xrightarrow{F} \mathcal{Y}$ is representable if and only if for every morphism $V \xrightarrow{\phi} U$ in (Aff/S) and every $y \in ob\mathcal{Y}_U$, the category $\mathcal{X}_{y,V}$ the objects of which are the pairs (x,g) with $x \in \mathcal{X}_V$ and $g \in \mathrm{Hom}_{\mathcal{Y}_V}(F(x), \phi^* y)$ and the morphisms of which are the pairs from (x_1, g_1) to (x_2, g_2) are the morphisms

$$x_1 \xrightarrow{h} x_2$$

in $\mathcal{X}_V$ such that $g_1 = g_2 \circ F(h)$ is equivalent to a discrete category Z_ϕ and if the functor

$$(\mathrm{Aff}/S) \rightarrow (Set)$$

$$(V \xrightarrow{\phi} U) \mapsto obZ_\phi$$

is representable by an algebraic U-space Z.

Definition 5.4 ([La2]). A local construction $\underline{\mathcal{Y}}$ over $\mathcal{X}$ is said to be algebraic (resp. algebraic and having a property P) if for every $U \in ob(\mathrm{Aff}/S)$ and every $x \in ob\mathcal{X}_U$, the U-champ $\underline{\mathcal{Y}}_U(x)$ is algebraic (resp. algebraic and the structure morphism $\underline{\mathcal{Y}}_U(x) \xrightarrow{F} U$ has the property P).

Proposition 5.3. If a local construction $\underline{\mathcal{Y}}$ over $\mathcal{X}$ which is algebraic (resp. algebraic and has the property P) over $\mathcal{X}$, the 1-morphism of S-champs $\mathcal{Y} \xrightarrow{F} \mathcal{X}$ is algebraic (resp. algebraic and has the property P).

Definition 5.5 ([Gir],[La2]). Let X be an algebraic S-space. The S-groupoid $Coh_{X/S}$ is defined as follows:
for every $U \in \mathrm{ob}(\mathrm{Aff}/S)$ the objects of the category of fibre over U the coherent $\mathcal{O}_{X\times_S U}$-Modules which are $\mathcal{O}_U$-flat and the morphisms are isomorphisms of $\mathcal{O}_{X\times_S U}$-Modules. We denote by $Fib^n_{X/S}$ an open fullsubchamps of $Coh_{X/S}$ whose objects of $Fib^n_{X/S\ U}$ consisting of $\mathcal{O}_{X\times_S U}$-Modules which are locally free of rank n.

Theorem 5.4 ([La2]). Let S be nötherian and $f : X \rightarrow S$ a projective scheme over S. Suppose for any base change $S' \rightarrow S$, $f_{S'}(\mathcal{O}_{X_{S'}}) = \mathcal{O}_{S'}$. Then $Coh_{X/S}$ is an algebraic S-champ locally of finite type and $Fib^n_{X/S}$ is an algebraic S-champ of finite type.

Hence we obtain the following theorem.

Theorem 5.5. Let S be nötherian normal and $f : X \rightarrow S$ a geometrically connected projective smooth scheme over S. Let $(D_i)_{i\in I}$ be a set of regular divisors with normal crossings only. An X-groupoid $\mathcal{Y}$ is defined as follows:
for every $U \in \mathrm{ob}(\mathrm{Aff})/X$ the objects of $\mathcal{Y}_U$ are $\mathcal{O}_U[(T_i)_{i\in I}]/((T_i^{n_i} - a_i)_{i\in I})$ for all sections a_i of $\mathcal{O}_U$ with $\mathrm{div}(a_i) = D_{iU}$ and the morphisms are isomorphisms of $\mathcal{O}_U$-Modules. Then $\mathcal{Y}$ is an open fullsubcategory of $Fib^n X/X$ for some n and $\mathcal{Y}$ is an algebraic X-champ.

6. Logarithmic structures and Applications

In differential geometric point of view, to obtain Kodaira's vanishing theorem, it is inevitable to generalize de Rahm cohomology to Kodaira-Dolbeault cohomology with values in vector bundles with Hermitian metrics: from flat structures to structures with curvature. In algebraic point of view, it is the theory of fractional logarithmic structures on algebraic champs which is a variation of logarithmic structures on schemes found by Fontaine-Illusie and Kato([Ka]).

Let $\mathcal{X}$ and $\mathcal{Y}$ be algebraic champs over S. The pre-logarithmic structures on $\mathcal{X}$ and $\mathcal{Y}$ are sheaves of monoids M and N on the smooth sites $\mathcal{X}_{lis}$ and $\mathcal{Y}_{lis}$ endowed with homomorphisms $\alpha : M \to \mathcal{O}_{\mathcal{X}}$ and $\beta : N \to \mathcal{O}_{\mathcal{Y}}$, respectively. One denotes the logarithmic differential sheaf by

$$\Omega^1_{(\mathcal{X},M)/(\mathcal{Y},N)} := (\Omega^1_{\mathcal{X}/\mathcal{Y}} \oplus \mathcal{O}_{\mathcal{X}} \otimes_{\mathbb{Z}} M^{gr})/\sim$$

where $\sim$ is defined by the $\mathcal{O}_{\mathcal{X}}$-submodule generated locally by local sections in the following form:

(1) $(d\alpha(a), 0) - (0, \alpha(a) \otimes a)$ with $a \in M$
(2) $(0, 1 \otimes a)$ with $a \in \mathrm{Im}(f^{-1}(N) \to M)$.

Theorem 6.1 (Kato's criterion([Ka])). Let $f : (X, M) \to (Y, N)$ be a morphism of schemes with fine logarithmic structures. Assume we are given a chart $Q_Y \to N$ of N. Then the following two conditions are equivalent.

(1) f is smooth(resp. etale).
(2) Etale locally on X, there exists a chart $(P_X \to M,\ Q_Y \to N,\ Q \to P)$ of f extending the given $Q_Y \to N$ satisfying the following two conditions;
 (i) $Ker(Q^{gp} \to P^{gp})$ and the torsion part of the cokernel $Coker(Q^{gp} \to P^{gp})_{tor}$ (resp. $Coker(Q^{gp} \to P^{gp})$) are finite groups of orders invertible on X.
 (ii) the induced morphism $X \to Y \times_{\mathrm{Spec}(\mathbb{Z}[Q])} \mathrm{Spec}(\mathbb{Z}[P])$ is etale in the sense of schemes.

Let $D = \sum_i r_i D_i$ $(r_i \in \mathbb{Q})$ be the irreducible decomposition and let $[D]$ denote $\sum_i [r_i] D_i$.

Theorem 6.2 (Deligne-Illusie([DI], [I], [Ka])). Let p be a prime number and $f : (\mathcal{X},\ M_0) \to (\mathcal{Y},\ N)$ a smooth morphism of algebraic champs with fine logarithmic structures over $S = \mathrm{Spec}(\mathbb{F}_p)$. Let $f' : (\mathcal{X}',\ M') \to (\mathcal{Y},\ N)$ be the base change of f by the absolute Frobenius map $F_{(\mathcal{Y},\ N)} \to F_{(\mathcal{Y},\ N)}$, let

$$(\mathcal{X},\ M) \xrightarrow{F} (\mathcal{X},\ M)$$

be the factorization of $F_{(\mathcal{X},\ M)}$ with $f = f' \circ F$ and let

$$(\mathcal{X},\ M) \xrightarrow{g} (\mathcal{X}'',\ M'') \xrightarrow{h} (\mathcal{X}',\ M')^{int}$$

be the factorization of

$$F^{int} : (\mathcal{X},\ M) \xrightarrow{g} (\mathcal{X}',\ M')^{int}$$

where "weakly radical" = "etale"$\circ$ "radical(i.e., exact)" with logarithmic structures. Let L be a fractional pre-logarithmic structure such that

(1) Locally on $\mathcal{X}_{lis}$, let P be a chart of M.
(2) Since P is a finitely generated monoid, there exists a surjection

$$\oplus_{i \in I} \mathbb{N} \to P$$

(3) Given a set of positive integers $(n_i)_{i\in I}$, there exists a push out R such that

$$\begin{tikzcd} \oplus_{i\in I}\mathbb{N} \arrow[r] \arrow[d, "nat."'] & P \arrow[d] \\ \oplus_{i\in I}\frac{1}{n_i}\mathbb{N} \arrow[r] & R \end{tikzcd}$$

(4) Locally on $\mathcal{X}_{lis}$, there exist sections $(a_i)_{i\in I}$ of $\mathcal{O}_{\mathcal{X}}$ such that

$$\begin{array}{l} \oplus_i\mathbb{N} \to P \to \mathcal{O}_{\mathcal{X}} \\ (k_i) \mapsto \qquad \prod_i a_i^{k_i} \end{array}$$

(5) Let $\mathcal{Z} = \operatorname{Spec} \mathcal{O}_{\mathcal{X}}[(T_i)_{i\in I}]/(T_i^{n_i} - a_i)_{i\in I}$ and $L = R\mathcal{O}^*_{\mathcal{Z}}$.

Let $\pi : \mathcal{Z} \to \mathcal{X}$ be the structure morphism of algebraic champs. Let R_0 be the monoid defined by the fibre product

$$\begin{tikzcd} \pi_*R \arrow[r] & \pi_*\mathcal{O}_{\mathcal{Z}} \\ R_0 \arrow[r] \arrow[u] & \mathcal{O}_{\mathcal{X}} \arrow[u] \end{tikzcd}$$

and $L_0 = R_0\mathcal{O}^*_{\mathcal{X}}$.

Assume that the following divisor D is given to be numerically equivalent to zero;

$$D = \sum_{i\in I}\frac{\nu_i}{n_i}\operatorname{div}(a_i) \equiv 0 \quad (\text{for } \nu_i \in \mathbb{Z}).$$

(1) Assume f is smooth. Consider the composite

$$s : (\mathcal{X},\ M) \xrightarrow{g} (\mathcal{X}'',\ M'') \xrightarrow{h} (\mathcal{X}',\ M')^{int} \longrightarrow (\mathcal{X}',\ M').$$

Then one has the canonical isomorphism of $\mathcal{O}_{\mathcal{X}''}$-modules

$$C^{-1} : \Omega^a_{(\mathcal{X}'',\ L''_0)/(\mathcal{Y},\ N)}([D'']) \to \underline{H}^a(\Omega^*_{(\mathcal{X},\ L_0)/(\mathcal{Y},\ N)}([D]))$$

for any $a \in \mathbb{Z}$ defined by

$$C^{-1}(\alpha \cdot \wedge_{1\le i\le a}\operatorname{dlog}(s^*m_i)) = g^*(\alpha) \wedge_{1\le i\le a} \operatorname{dlog}(m_i)$$

$(\alpha \in \mathcal{O}_{\mathcal{X}''},\ m_i \in M)$.

(2) Assume that f is smooth and integral and that there exists an algebraic champ $(\tilde{\mathcal{Y}},\ \widetilde{N})$ with a fine log. str. such that $\tilde{\mathcal{Y}}$ is flat over $\operatorname{Spec}(\mathbb{Z}/p^2\mathbb{Z})$ and an isomorphism

$$(\mathcal{Y},\ N) \cong (\tilde{\mathcal{Y}},\ \widetilde{N}) \times_{\operatorname{Spec}(\mathbb{Z}/p^2\mathbb{Z})} \operatorname{Spec}(\mathbb{F}_p)$$

where $\operatorname{Spec}(\mathbb{Z}/p^2\mathbb{Z})$ and $\operatorname{Spec}(\mathbb{F}_p)$ are endowed with trivial log. str.'s.
Then there exists the canonical biunivoque between the set of isomorphism classes of the smooth lifting of $(\mathcal{X}'',\ L''_0)$ over $(\tilde{\mathcal{Y}},\ \widetilde{N})$ and the set of the splitting of $\tau_{<1}F_*\Omega^*_{(\mathcal{X},\ L_0)/(\mathcal{Y},\ N)}([D])$ in the derived category of $\mathcal{O}_{\mathcal{X}''}$-modules. If a smooth lifting of $(\mathcal{X},\ L_0)$ over $(\tilde{\mathcal{Y}},\ \widetilde{N})$, there exists an isomorphism

$$\tau_{<p}F_*\Omega^*_{(\mathcal{X},\ L_0)/(\mathcal{Y},\ N)}([D]) \cong \oplus_{0\le a<p}\Omega^a_{(\mathcal{X}'',\ L''_0)/(\mathcal{Y},\ N)}([D''])[-a]$$

in the derived category of $\mathcal{O}_{\mathcal{X}''}$-modules.

(3) Assume the following three conditions

(i) f is smooth and of Cartier type.
(ii) the underlying morphism of algebraic champs $\mathcal{X} \to \mathcal{Y}$ is proper.
(iii) Suppose locally on $\mathcal{Y}_{lis}$, there exist $(\widetilde{Y}, \widetilde{N})$ and a smooth lifting of $(\mathcal{X}', M')$ over $(\widetilde{Y}, \widetilde{N})$.

Then, the Hodge spectral sequence

$$E_1^{a,b} = R^b f_* \Omega^a_{(\mathcal{X},\ L_0)/(\mathcal{Y},\ N)}([D]) \Rightarrow R^{a+b} f_* \Omega^*_{(\mathcal{X},\ L_0)/(\mathcal{Y},\ N)}([D])$$

degenerates at $E_1^{a,b}$ for $a+b < p$.
The $\mathcal{O}_{\mathcal{Y}}$-modules $R^b f_* \Omega^a_{(\mathcal{X},\ L_0)/(\mathcal{Y},\ N)}([D])$ for $a+b < p$ are locally free and commute with any base change.

7. Purity

The notion of algebraic champs enables us to satisfy the assumption of Serre's Theorems of Kähler analogue of certain conjectures of Weil.

Theorem 7.1 (cf.[Ser]**).** Let S be Spec $(\mathbb{C})$. Let $\mathcal{Z}$ be a projective irreducible reduced non singular algebraic champ over S and $f : \mathcal{Z} \to \mathcal{Z}$ a morphism of algebraic champs. Suppose that there exists an integer $q > 0$ and a hyperplane section $\mathcal{H}$ of $\mathcal{Z}$ such that $f^{-1}(\mathcal{H})$ is algebraically equivalent to $q\mathcal{H}$. Then the absolute value of the eigen value of the endomorphism $f_n^* : H^n_{\text{de Rahm}}(\mathcal{Z}) \to H^n_{\text{de Rahm}}(\mathcal{Z})$ induced by f is just $q^{\frac{n}{2}}$ for every integer $n \geq 0$.

Theorem 7.2. Let S be Spec $(\mathbb{C})$. Let $\mathcal{X}$ be a projective irreducible reduced non singular algebraic champ over S and L an ample invertible sheaf over $\mathcal{X}$. Suppose L^m has a section s such that $E = $ div (s) is non singular, where m is an arbitrary positive integer.. Let $\pi : \mathcal{Z} \to \mathcal{X}$ be an algebraic champ *Spec* $(\mathcal{O}_{\mathcal{X}}[T]/(T^m - s))$ for simplicity. Then for $q > 0$ one has a morphism f such that $f^*(\pi^* L) = \pi^* L^{\otimes q}$.

(1) if $m > 1$, one has the endomorphism f_n^* of $H^n(\mathcal{X}, \Omega^*_{\mathcal{X}}\langle E\rangle \otimes L^{-1})$.
(2) if $m = 1$, one has the endomorphism f_n^* of $H^n(\mathcal{X}, \Omega^*_{\mathcal{X}})$.

The absolute values of the endomorphisms f_n^* above are all $q^{\frac{n}{2}}$. Hence the perverse sheaves $\Omega^*_{\mathcal{X}}$ and $\Omega^*_{\mathcal{X}}\langle E\rangle \otimes L^{-1}$ are pure of weight 0.

REFERENCES

[DI] DELIGNE, P. AND ILLUSIE, L., *Relèvement modulo p^2 et décomposition du complexe de de rham*, Invent. Math. 89(1987), 247-270.

[Fal] FALTINGS, G., *p-adic Hodge theory*, J. Amer. Math. Soc. 1 (1988), 255-299.

[Fal2] ———, *Crystalline cohomology and p-adic Galois representations*, J. Algebraic Analysis, Geometry and Number Theory, The Johns Hopkins Univ. Press, Baltimore, MD, 1989, 25-80.

[Fuj] FUJITA, T., *Classification Theories of Polarized Varieties*, London Math. Soc. Lecture Note Series *155*, Cambridge Univ. Press(1990).

[Gir] GIRAUD, J., *Cohomologie non abeliénne*, Springer-Verlag, New York, 1971.

[Gros] GROS, M., *Classes de Chern et classes de cycles en cohomologie de Hodge-Witt logarithmique*, Bull. Soc. Math. France, Mémoire 21, 1985.

[Gr] GROTHENDIECK, A., *Standard conjectures on algebraic cycles*, Algebraic Geometry, Oxford University Press, London and New York, 1969, 193-199.

[GM] GROTHENDIECK, G. AND MURRE, J.P., *The tame fundamental group of a formal neighbourhood of a divisor with normal crossings on a scheme* VIII, pp.133, 1971.

[Sa] SAAVEDRA RIVANO, N., *Catégories Tannakiennes*, Lecture Notes in Math., vol.265, Springer-Verlag, Berlin and New York, 1972.

[I] ILLUSIE, L., *Réduction semi-stable et décomposition de complexes de de Rham à coefficients*, Duke Math. J. 60(1990), 139-185.

[Il] ———, *Logarithmic spaces, according to K. Kato*, Barsotti Memorial Symposium on Algebraic Geometry, Padova, 1991.

[Ka] KATO, K., *Logarithmic structures of Fontaine-Illusie*, Algebraic Analysys, Geometry and Number Theory, The John-Hopkins Univ. Press, Baltimore, MD, 1989, 191-224.

[La1] LAUMON, G., *Transformation de Fourier constantes d'équations fonctionelles et conjecture de Weil*, Pub. Math. I.H.E.S. 65 (1987), 131-210.

[La2] ———, *Champs algébrique*, Prépublication de l'Université Paris-Sud(1988)

[Ser] SERRE, J.-P., *Analogues kählériens de certaines conjectures de Weil*, Ann.of Math. (2) 71, 1960, 392-394.

[Ta] TATE, J., *Conjectures on Algebraic Cycles in ℓ-adic Cohomology*, Proceeding of Symposia in Pure Mathematics, Vol. 55(1994), Part 1.