FUJITA CONJECTURE AND NUMERICAL EQUIVALENCE

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ABSTRACT. In this article the authors show the proofs of conjectures formulated by Fujita ([6]) and propose to adopt numerical equivalence classes of divisors in place of linear equivalence classes when classifying algebraic varieties, since Enault-Viehweg type vanishing theorems([11]) hold up to numerical equivalence. Zariski decomposition are treated in numerical equivalence. One of the main concerns in birational geometry is whether the amplitude of the variation of fibres should be bounded above by the difference between the relative Kodaira dimensions of the whole and the generic fibre([10], [1]). This theme can be transferred to new relative Kodaira dimensions defined by numerical equivalence classes([9]).

1. INTRODUCTION

Let X be a projective non singular variety of dimension d over the complex number field $\mathbb{C}$, K a canonical divisor and D an ample divisor on X, respectively. By Mori theory([1], [10]), $K + mD$ is nef if $m > d$. If $K + mD$ is nef for a positive number m, $n(K + mD)$ is spanned (i.e., very abundant) for a sufficiently large number n by the base point free theorem. We would estimate this number n. Fujita([4], [5], [8], [14]) has researched adjoint line bundles and many examples to pose the following conjectures([6]):

Conjecture 1 (Fujita conjecture A). $K + mD$ is very abundant if

(1) $m > d$,
(2) $m = d$ and the self-intersection number $I(D) = D^d > 1$.

Conjecture 2 (Fujita conjecture B). $K + mD$ is very ample if

(1) $m > d + 1$
(2) $m = d + 1$ and the self-intersection number $I(D) = D^d > 1$.

The authors will give proofs of conjectures above in Section 2. Fujita proved these conjectures when $d \leq 3$, using Reider's theory and results of Ein-Lazarsfeld's. These bounds are best possible, since there exists a Del Pezzo manifold (X, D) with $I(D) = 1$ such that $K + d\,D = D$ is not spanned and $K + (d+1)\,D = 2D$ is not very ample([7]). We define an analogue of Iitaka dimension:

Definition 1.1. $\iota(D) = \max\{\kappa(D') \mid D' \text{ is numerically equivalent to } D\}$

We pose a question for a fibre space $f : X \to S$:

Conjecture 3.

$$\iota(K_{X/S}) \geq \iota(K_{(X/S)_{\bar{\eta}}}) + \mathrm{var}(X/S),$$

where $\bar{\eta}$ is the separable closure of the generic point of S. If the automorphisms of local monodromies of $R^d f_* \mathbb{C}$ are unipotent, the equality holds.

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2. Preliminaries

Notation 2.1. We denote by $\mathrm{Div}(X)$ the divisor group of X.

A divisor D is called to be very abundant(resp. abundant) if D(resp. mD for some $m > 0$) is linearly equivalent to the pull-back of a very ample divisor by some morphism. We call a divisor D is numerically abundant, if D is numerically equivalent to an abundant divisor. The complete linear system associated to D is said to be free, if it has no base point. In other word, $\mathcal{O}(D)$ is spanned by the global sections.

Lemma 2.1. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms and $\mathcal{L}$ an invertible sheaf. Suppose $g^*g_*(\mathcal{L}) \to \mathcal{L}$ is surjective. Then $f^*g^*g_*f_*f^*(\mathcal{L}) \to f^*(\mathcal{L})$ is also surjective.

Proof.
$$f^*g^*g_*(\mathcal{L}) \to f^*\mathcal{L}$$
is surjective. From $id \to f_*f^*(\mathcal{L})$, this morphism factors through
$$f^*g^*g_*f_*f^*(\mathcal{L}) \to f^*(\mathcal{L}).$$
□

Proposition 2.2. The following conditions are equivalent

(1) A divisor D is very abundant.
(2) The complete linear system $|D|$ is free.

Proof. If the canonical homomorphism $\mathcal{O} \otimes H^0(X, \mathcal{O}(D)) \to \mathcal{O}(D)$ is surjective, $\mathcal{O}(D)$ is the pull-back of a very ample invertible sheaf.
□

Theorem 2.3. The sum of two abundant divisors is abundant.

Proof. The external tensor product of two ample invertible sheaves over two schemes is ample. □

Proposition 2.4. Let $\mathcal{L}$ be an invertible sheaf over a scheme. The following conditions are equivalent

(1) $\mathcal{L}^{\otimes n}$ is very abundant for some $n > 0$.
(2) $\mathcal{L}^{\otimes n}$ is very abundant for every $n \geq n_0$ for some n_0.

Proof. The statement is valid for very ampleness. Hence one gets the proof. □

Proposition 2.5. Let X be a scheme, $\mathcal{L}$ a very ample invertible sheaf and $\mathcal{K}$ a very abundant invertible sheaf on X. Then $\mathcal{L} \otimes \mathcal{K}$ is also very ample.

Proof. By assumption, there exist morphisms $f : X \to Y$ and $g : X \to Z$ and very ample invertible sheaves $\mathcal{O}_Y(1)$, $\mathcal{O}_Z(1)$ on Y and Z such that $\mathcal{L}$ and $\mathcal{K}$ are the pull-backs of $\mathcal{O}_Y(1)$ and $\mathcal{O}_Z(1)$, respectively. We have an immersion $(f,g) : X \to Y \times Z$ such that $\mathcal{L} \otimes \mathcal{K} \cong (f,g)^*\mathcal{O}_Y(1) \otimes \mathcal{O}_Z(1)$. □

We give another interpretation of a criterion of very-ampleness in case of complex varieties.

Proposition 2.6. Let X be a projective non singular variety over the complex number field $\mathbb{C}$. The sum of a very ample divisor H and a very abundant divisor D on X is very ample.

Proof. Note that if H is very ample, $H^1(X,\mathcal{O}(H)(-x-y)) \cong H^1(X,\mathcal{O}(H))$, where $x,y \in X$. Let H be a very ample non singular divisor and D a non singular very abundant divisor. We proceed by induction of dimension. Since the following sequence is exact;

$$\begin{array}{ccccc} H^1(X,\mathcal{O}_X(H)(-x-y)) & \to & H^1(X,\mathcal{O}_X(H+D)(-x-y)) & \to & H^1(D,\mathcal{O}_D(H+D)(-x-y)) \\ \downarrow & & \downarrow & & \downarrow \\ H^1(X,\mathcal{O}_X(H)) & \to & H^1(X,\mathcal{O}_X(H+D)) & \to & H^1(D,\mathcal{O}_D(H+D)) \end{array}$$

where D can be through x,y, if necessary, replaced by a suitable one, isomorphisms of both side terms implies the isomorphism of $H^1(X,\mathcal{O}_X(H+D)(-x-y)) \to H^1(X,\mathcal{O}_X(H+D))$. □

Theorem 2.7 ([3], [11], [13]). Let X be a complete non singular variety over $\mathbb{C}$ and $D \in Div(X) \otimes \mathbb{R}$. Assume D is numerically zero and the fractional part of D is supported in a normal crossing divisor. Let C be a non singular divisor contained in the support of the fractional part of D.

Then

$$H^b(X,\Omega^a_X\langle\lceil\{D\}\rceil - C\rangle(C+[D])) \to H^b(X,\Omega^a_X\langle\lceil\{D\}\rceil\rangle(C+[D]))$$

are injective for all a,b. In particular,

$$H^b(X,\omega_X(\lceil D\rceil)) \to H^b(X,\omega_X(C+\lceil D\rceil))$$

are injective. In addition, let E be a divisor on X. Assume $\lceil\{D\}\rceil + E$ is normal crossing. Then

$$H^b(X,\Omega^a_X\langle E+\lceil\{D\}\rceil - C\rangle(C+[D])) \to H^b(X,\Omega^a_X\langle E+\lceil\{D\}\rceil\rangle(C+[D]))$$

are injective for all a,b. In particular,

$$H^b(X,\omega_X(E+\lceil D\rceil)) \to H^b(X,\omega_X(C+E+\lceil D\rceil))$$

are injective.

Proposition 2.8. Let X be a complete non singular variety and D a divisor on X. The sheaf of effective divisors numerically equivalent to D is $\oplus_{\mathcal{O}(C)\in \mathrm{Pic}^\tau(X)}\mathcal{O}(D+C)$. For example the sum of the complete linear system of the divisors numerically equivalent to D is

$$\oplus_{\mathcal{O}(C)\in \mathrm{Pic}^\tau(X)}H^0(X,\mathcal{O}(D+C)).$$

3. Theorems and Proofs

Theorem 3.1 (Fujita Conjecture A). Let X be a complete non singular variety of dimension d over $\mathbb{C}$ and $\mathcal{L}$ (resp. $(\mathcal{L}_i)_{1\le i\le d}$) an invertible sheaf(resp. invertible sheaves) over X.

Suppose that

(1) $\mathcal{L}$ is ample(resp. the $\mathcal{L}_i$ are ample).

(2) the self intersection number $I(\mathcal{L}) > 1$(resp. the intersection number $(\mathcal{L}_1,\cdots,\mathcal{L}_d) > 1$).

Then $\omega_X \otimes \mathcal{L}^{\otimes m}$(resp. $\omega_X \otimes \otimes_{1\le i\le d}\mathcal{L}_i$) is abundant. In other words,

$$Bs|\omega_X \otimes \mathcal{L}^{\otimes m}| = \emptyset \qquad \text{for } m \ge \dim X$$

$$(\text{resp. } Bs|\omega_X \otimes \otimes_i\mathcal{L}_i| = \emptyset.)$$

Lemma 3.2. Assume moreover. Let X be a complete non singular variety over $\mathbb{C}$ and let $\mathcal{L} = \mathcal{O}(H)$(resp. $\mathcal{L}_j = \mathcal{O}(H)$) by a non singular hyperplane section H of X.

Then $\omega_X \otimes \mathcal{L}^{\otimes m}$ (resp. $\omega_X \otimes \otimes_{1 \leq i \leq d} \mathcal{L}_i$) is generated by the global sections.

In other words,

$$Bs|\omega_X(mH)| = \emptyset \qquad \text{for } m \geq \dim X$$

$$(\text{resp. } Bs|\omega_X \otimes \otimes_i \mathcal{L}_i| = \emptyset.)$$

Proof. We proceed by induction. If dim $X = 1$, $\omega_X(H)$ is abundant since deg $H > 1$. Assume dim$X \geq 2$. If dim $\mathrm{Bs}|\omega_X(mH)| \geq 1$ for $m \geq$ dim$X$ (resp. dim $\mathrm{Bs}|\omega_X(H) \otimes \otimes_{i \neq j} \mathcal{L}_i| \geq 1$) then we have a base point on H. Since $H^0(X, \omega_X(mH)) \to H^0(H, \omega_H((m-1)H)) \to 0$ (resp. $H^0(X, \omega_X \otimes \otimes_{1 \leq i \leq d} \mathcal{L}_i) \to H^0(H, \omega_H \otimes \otimes_{i \neq j} \mathcal{L}_i) \to 0$) is exact by Kodaira vanishing $H^1(X, \omega_X((m-1)H)) = 0$ (resp. $H^1(X, \otimes \otimes_{i \neq j} \mathcal{L}_i) = 0$) and $|\omega_H((m-1)H))|$ (resp. $\omega_H \otimes \otimes_{i \neq j} \mathcal{L}_i$) is free by induction assumption. It is a contradiction. If dim $\mathrm{Bs}|\omega_X(mH)| = 0$ for $m \geq$ dim$X$ (resp. dim $\mathrm{Bs}|\omega_X(H) \otimes \otimes_{i \neq j} \mathcal{L}_i| \geq 0$) , then take a hyperplane through a base point. We have a contradiction. □

Proof. (Proof of Theorem(Fujita Conjecture A)) Assume dim $X > 1$. Assume that the k-th power of an ample invertible sheaf $\mathcal{L}$(resp. an ample invertible sheaves $\mathcal{L}_j$) for $k \geq 2$ has a non singular irreducible hyperplane section H. Take a k-cyclic covering totally ramified along H, denote it by $f : Y \to X$. We denote f^*H by H_Y. Note that H_Y is divisible by k and write it by D, which is non singular and irreducible. Let $m \geq d$. The natural homomorphism

$$H^0(Y, \omega_Y(mD)) \to H^0(D, \omega_D((m-1)D))$$

(resp. $H^0(Y, \omega_Y(D) \otimes \otimes_{i \neq j} f^*\mathcal{L}_i) \to H^0(D, \omega_D \otimes \otimes_{i \neq j} \mathcal{L}_i)$ is surjective, since $H^1(Y, \omega_Y((m-1)D)) = 0$(resp. $H^1(Y, \omega_Y \otimes \otimes_{i \neq j} f^*\mathcal{L}_i) = 0$.) By induction assumption, $|\omega_D((m-1)D)|$ (resp. $|\omega_Y(D) \otimes \otimes_{i \neq j} f^*\mathcal{L}_i|$) is free.

Note that $\mathcal{L}_Y \equiv \mathcal{O}_Y(D)$.

On the other hand, $H^0(X, \omega_X \otimes \mathcal{L}^{\otimes m})$(resp. $H^0(X, \omega_X \otimes \otimes_{1 \leq i \leq d} \mathcal{L}_i)$) is the Galois invariant part([12]) by Gal(Y/X) of $H^0(Y, \omega_Y \otimes f^*\mathcal{L}_Y^{\otimes m})$ (resp. $H^0(Y, \omega_Y \otimes \omega_Y \otimes_{1 \leq i \leq d} f^*\mathcal{L}_i)$). To each member M of $|\omega_D \otimes \mathcal{L}_Y^{\otimes m-1}|$ (resp. $|\omega_D \otimes \otimes_{i \neq j} \mathcal{L}_i|$), there corresponds a divisor W of $|\omega_X \otimes \mathcal{L}^{\otimes m}|$ (resp. $|\omega_Y \otimes \otimes_{1 \leq i \leq d} f^*\mathcal{L}_i|$) which is an extension. The restriction of $\frac{1}{k} \sum_{\sigma \in \mathrm{Gal}(Y/X)} W^\sigma \in |\omega_X \otimes \mathcal{L}^{\otimes m}|$ (resp. $\frac{1}{k} \sum_{\sigma \in Gal(Y/X)} W^\sigma \in |\omega_X \otimes \otimes_i \mathcal{L}_i|$) to D is M. Thus the natural homomorphism $H^0(X, \omega_X \otimes \mathcal{L}^{\otimes m}) \to H^0(D, \omega_D \otimes \mathcal{L}_Y^{\otimes m-1})$ (resp. $H^0(X, \omega_X \otimes \otimes_i \mathcal{L}_i) \to H^0(D, \omega_D \otimes \mathcal{L}_Y^{\otimes m-1})$) is surjective. Therefore the complete linear system $|\omega_X \otimes \mathcal{L}^{\otimes m}|$(resp. $|\omega_X \otimes \otimes_i \mathcal{L}_i|$) has no base point on H. Moving H, we conclude that $|\omega_X \otimes \mathcal{L}^{\otimes m}|$ (resp. $|\omega_X \otimes \otimes_i \mathcal{L}_i|$) has no base point on the whole X. □

Theorem 3.3 (Fujita Conjecture B). Let X be a complete non singular variety of dimension d over $\mathbb{C}$ and $\mathcal{L}$ an ample invertible sheaf(resp. $\mathcal{L}_i$ ample invertible sheaves) over X. Then if $m \geq d+1$, $\omega_X \otimes \mathcal{L}^{\otimes m}$ (resp. $\omega_X \otimes \otimes_{1 \leq i \leq d+1} \mathcal{L}_i$) is very ample when the self-intersection $I(\mathcal{L}) > 1$ (resp. $(\mathcal{L}_{i_1}, \cdots, \mathcal{L}_{i_d}) > 1$ for any $\{i_1, \cdots, i_d\} \subset \{1, \cdots, d+1\}$).

Lemma 3.4. Assume moreover. Let X be a complete non singular variety over $\mathbb{C}$ and H a very ample non singular divisor on X such $\mathcal{L} = \mathcal{O}(H)$ (resp. $\mathcal{L}_j = \mathcal{O}(H)$). Then if $m \geq$ dim $X + 1$, the invertible sheaf $\omega_X(mH)$(resp. $\omega_X \otimes \otimes_{1 \leq i \leq d+1} \mathcal{L}_i$) is very ample when the self-intersection $I(\mathcal{H}) > 1$(resp. $(\mathcal{L}_{i_1}, \cdots, \mathcal{L}_{i_d}) > 1$ for any $\{i_1, \cdots, i_d\} \subset \{1, \cdots, d+1\}$.)

Proof. Let $m > d+1$. The next exact sequence

$$H^0(X, \omega_X(mH)) \to H^0(H, \omega_H((m-1)H)) \to 0$$

(resp. $H^0(X, \omega_X(H) \otimes \otimes_{i\neq j}\mathcal{L}_i) \to H^0(H, \omega_H \otimes \otimes_{i\neq j}\mathcal{L}_i))$ implies the commutativity of the following square

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{P}(H^0(X, \omega_X(mH))) \\ \uparrow & & \uparrow \\ H & \longrightarrow & \mathbb{P}(H^0(H, \omega_H((m-1)H))). \end{array} \tag{3.1}$$

(resp.

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{P}(H^0(X, \omega_X(H) \otimes \otimes_{i\neq j}\mathcal{L}_i)) \\ \uparrow & & \uparrow \\ H & \longrightarrow & \mathbb{P}(H^0(H, \omega_H(H) \otimes \otimes_{i\neq j}\mathcal{L}_i)) . \end{array} \tag{3.2}$$

Thus by moving H, we get the proof. $\square$

Proof. (Proof of Theorem(Fujita Conjecture B)) Let $m > d+1$ and $I(\mathcal{L}) > 1$ (resp. $(\mathcal{L}_{i_1}, \cdots, \mathcal{L}_{i_d}) > 1$.) The divisor $\omega_C \otimes \mathcal{L}^{\otimes m}$(resp. $\omega_C \otimes \otimes_i \mathcal{L}_i$) is very ample on a curve C if deg $D > 1$ and $m - 1 > \dim C = 1$. We proceed by induction. For some $k > 0$, we have $\mathcal{L}^{\otimes k} = \mathcal{O}_X(H)$(resp. $\mathcal{L}_j^{\otimes k} = \mathcal{O}_X(H)$), where H is very ample and non singular. Construct a k-cyclic cover $f : Y \to X$ totally ramified at H over X. $H_Y = f^*H$ is divisible by k and we denote it by D. Since $\mathcal{L}_Y \equiv \mathcal{O}_Y(D)$, one has a surjection

$$H^0(Y, \omega_Y(D) \otimes \mathcal{L}_Y^{\otimes m-1}) \to H^0(D, \omega_D \otimes \mathcal{L}_Y^{\otimes m-1})$$

(resp. $H^0(Y, \omega_Y(D) \otimes \otimes_{i\neq j}\mathcal{L}_i) \to H^0(H, \omega_H \otimes \otimes_{i\neq j}\mathcal{L}_i)$.) We have thus a commutative square;

$$\begin{array}{ccc} Y & \longrightarrow & \mathbb{P}(H^0(Y, \omega_Y(mD))) \\ \uparrow & & \uparrow \\ D & \longrightarrow & \mathbb{P}(H^0(D, \omega_D((m-1)D))). \end{array}$$

(resp.

$$\begin{array}{ccc} Y & \longrightarrow & \mathbb{P}(H^0(Y, \omega_Y(D) \otimes \otimes_{i\neq j}\mathcal{L}_i)) \\ \uparrow & & \uparrow \\ D & \longrightarrow & \mathbb{P}(H^0(D, \omega_D \otimes \otimes_{i\neq j}\mathcal{L}_i)) . \end{array}$$

Since the natural homomorphism $H^0(X, \omega_X \otimes \mathcal{L}^{\otimes m}) \to H^0(D, \omega_D \otimes \mathcal{L}_Y^{\otimes m-1})$ (resp. $H^0(X, \omega_X \otimes \otimes_i \mathcal{L}_i) \to H^0(D, \omega_D \otimes \mathcal{L}_Y^{\otimes m-1})$) is surjective(Proof of Theorem(3.1)(Fujita Conjecture A)), one has a commutative square;

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{P}(H^0(X, \omega_X \otimes \mathcal{L}^{\otimes m})) \\ \uparrow & & \uparrow \\ H & \longrightarrow & \mathbb{P}(H^0(D, \omega_D \otimes \mathcal{L}^{\otimes m-1})). \end{array}$$

(resp.

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{P}(H^0(X, \omega_X \otimes \otimes_{1\leq i\leq d+1}\mathcal{L}_i)) \\ \uparrow & & \uparrow \\ H & \longrightarrow & \mathbb{P}(H^0(D, \omega_D \otimes \otimes_{i\neq j}\mathcal{L}_i)) .) \end{array}$$

Moving H, we get the proof. $\square$

4. Cone type Theorem

Theorem 4.1 (Cone type Theorem). Let X be a complete non singular variety over $\mathbb{C}$, K a canonical divisor on X and H a divisor.

(C) Assume $\mathcal{O}(K+D)$ is zero in $\mathrm{Pic}^{\tau}(X)$, if $\mathcal{O}(K+D) \in \mathrm{Pic}^{\tau}(X)$, where D is a normal crossing divisor. Suppose

(1) For a number a, H and $aH - K$ are numerically abundant, respectively.
(2) Some multiple $b > 0$ of H contains a canonical divisor K, i.e., $bH - K$ is effective .

Then H is also abundant.

Proof. An abundant divisor has non zero Iitaka dimension we denote by $\kappa \geq 0$.

(i) When H and $aH - K$ are numerically zero, it is obvious by hypothesis.

Suppose $aH-K$ for a number a is numerically equivalent to an abundant divisor of strictly positive Iitaka dimension $\kappa > 0$. This will be shown later. $\ell H - K$ for $0 << \ell$ is linearly equivalent to an abundant divisor. By hypothesis C, H is also abundant. Suppose H is numerically equivalent to an abundant divisor of strictly positive Iitaka dimension $\kappa > 0$. We proceed by induction on dimension of varieties. If $\dim(X) = 1$, it is easy because H is ample because the genus of X is non zero or $\mathrm{Pic}^{\tau}(\mathbb{P}^1) = 0$. Assume $\dim(X) > 1$. Since Iitaka dimension of an abundant divisor which is numerically equivalent to $bH - K$ is greater than one for infinitely many $b > 0$ by assumption, there exist a non singular abundant divisor D' and a number m such that $m(bH - K) - D'$ is numerically equivalent to zero. We have also a non singular abundant divisor D such that $D + D'$ is normal crossing and $nH - D$ is numerically equivalent to zero for a number n. There exists a non singular abundant divisor D'' such that $k(2nH - D) - (D + D'')$ is numerically equivalent to zero for a number k and $D + D' + D''$ is normal crossing.

One has the following exact sequence;

$$0 \to \mathcal{O}(K + \lceil bH - K - \epsilon' D' + (2nH - D) - \epsilon(D + D'')\rceil) \to$$
$$\mathcal{O}(K + D + \lceil bH - K - \epsilon' D' + (2nH - D) - \epsilon(D + D'')\rceil) \to$$
$$\mathcal{O}_D(K + D\lceil bH - K - \epsilon' D' + (2nH - D) - \epsilon(D + D'')\rceil) \to 0$$

where $\epsilon = \frac{1}{k}$, $\epsilon' = \frac{1}{m}$. The next long exact sequence is derived;

$$0 \to H^0(X, \mathcal{O}((2n+b)H - D)) \to H^0(X, \mathcal{O}((2n+b)H)) \to H^0(D, \mathcal{O}_D((2n+b)H)) \to \cdots .$$

By Theorem(2.7), $H^1(X, \mathcal{O}((2n+b)H - D)) \to H^1(X, \mathcal{O}((2n+b)H))$ is injective. Thus one concludes that there exists no base point on the subset D in X of $(2n+b)H$. The family of non singular divisor linearly equivalent to D go through every point in X. Hence the complete linear system of $(2n+b)H$ is free. Therefore H is abundant. $\square$

Remark 4.1. The fundamental inequality $\kappa(\omega_{X/S}\langle D\rangle) \geq \kappa(\omega_{X_\eta}\langle D_\eta\rangle) + \mathrm{logvar}((X - D)/S)$ implies the assumption of Theorem(4.1). Its idea is originally due to Tsunoda; If the irregularity vanishes, there remains noting to prove. If otherwise, there exists the Albanese map whose image has non negative κ. One can proceed by induction on dimension.

5. Zariski Decomposition

We study an analogue of Zariski decomposition. We denote by $N^1(X)$ (resp. $N^1(X)_{\mathbb{Q}}$, resp.$N^1(X)_{\mathbb{R}}$) Div(X) modulo numerical equivalence(resp. $N^1(X) \otimes \mathbb{Q}$, resp. $N^1(X) \otimes \mathbb{R}$).

Definition 5.1. A divisor D is said to have a Zariski decomposition in Div$(X) \otimes \mathbb{Q}$(resp. Div$(X) \otimes \mathbb{R}$, resp. $N^1(X)_{\mathbb{Q}}$, resp. $N^1(X)_{\mathbb{R}}$), if

$$D \equiv P + N$$

such that

(1) P is abundant,
(2) P is a divisor which contains a canonical divisor in Pic$(X) \otimes \mathbb{R}$.
(3) N is an effective divisor,
(4) $H^0(X, \mathcal{O}([nD])) = H^0(X, \mathcal{O}([nP]))$ for every number n.

Theorem 5.1. Let X be a complete non singular variety over $\mathbb{C}$ and D an effective divisor. Assume (C) in Theorem(4.1).

Suppose a divisor $H \in$ Div(X) has a Zariski decomposition in $N^1(X)_{\mathbb{R}}$;

(1) the sum of all fractional divisors are supported in a normal crossing divisor.
(2) H is a divisor which contains a canonical divisor K in Pic$(X) \otimes \mathbb{R}$.

$$H \equiv P + N$$

(3) P is abundant,
(4) N is effective,
(5)

$$bH - K \equiv Q + M \qquad \text{for a number } b.$$

(6) Q is abundant.
(7) M is effective.
(8) M and N are strictly stable fixed component of H.

Then the divisor D has a Zariski decomposition in Div$(X) \otimes \mathbb{Q}$.

Proof. (i) Assume P is numerically zero. It reduces to assumption C (see Proof of Theorem(4.1).)

(ii) Assume P has strictly positive Iitaka dimension.

We can write

(1) $H \equiv \epsilon D + N$ for $0 < \epsilon << 1$.
Here D is non singular.
(2) $bH - K \equiv \epsilon' D' + M$ for $0 < \epsilon' << 1$. Here D' is non singular.

Hence one has

$$\begin{aligned} mH - D &\equiv mP + mN - D \\ &\equiv (m\epsilon - 1)D + mN. \end{aligned}$$

We choose m such that $1 < m\epsilon < 2$. Thus one has the next epimorphism by Theorem(2.7),

$$H^0(X, \mathcal{O}(K + D + \lceil mH - D - (m\epsilon - 1)D - mN + bH - K - \epsilon' D' - M \rceil)) =$$
$$H^0(X, \mathcal{O}((m+b)H - [mN + M])) \to H^0(D, \mathcal{O}_D((m+b)H - [mN + M])).$$

Since M and N are strictly stable component of H, we see $H^0(X, \mathcal{O}((m+b)H - [mN+M])) = H^0(X, \mathcal{O}((m+b)H))$ and $H^0(D, \mathcal{O}_D((m+b)H - [mN+M])) = H^0(D, \mathcal{O}_D((m+b)H))$. Therefore H has Zariski decomposition in Div$(X) \otimes \mathbb{Q}$. $\square$

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