

Flat deformation theory of a family dominated by another family

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Introduction.

We shall work over the field of complex numbers and prove the following theorem.

Theorem.

Let T be a manifold, X, Y, Z irreducible reduced T -schemes of finite type and $p : X \rightarrow T, q : Y \rightarrow T, r : Z \rightarrow T$ proper flat morphisms over T . Assume the next commutative diagram

$$\begin{array}{ccccc} & & X & & \\ f \swarrow & & & \searrow h & \\ Y & & p \downarrow & & Z \\ q \searrow & & & \swarrow r & \\ & & T & & \end{array}$$

Satisfying the following condition :

- (i) f, h are proper surjective morphisms,
- (ii) Y is biholomorphically trivial over T , i. e., $X \simeq Y_0 \times T$ for some complete variety Y_0 ,
- (iii) there exists a proper surjective morphism $g_t : Y_t \rightarrow Z_t$ with $h_t = g_t \circ f_t$ for each point t of T .

Then Z is biholomorphically trivial over a non-empty open subset T^0 of T .

Note that the theorem above holds in the category of analytic spaces.

It is well-known that there exists a family of uncountable birational varieties dominated by a fixed variety via rational mappings, say, a family of uncountable cubic threefolds, which are unirational and mutually birationally inequivalent. Applying the theorem to it, we conclude that we cannot resolve all the indeterminacies of the rational mappings of a fixed variety onto the varieties by means of birational transformations even in the category of reduced schemes of finite type.

§ 1. Preliminary and Result.

We recall some definitions and preliminary properties of the theory of deformation.

Let X be a scheme. An extension of X -Algebra is defined to be a commutative triangle of Algebras

$$\begin{array}{ccc} E & \xrightarrow{p} & B \\ \searrow & & \uparrow \\ & & A \end{array}$$

where p is an epimorphism such that $(\ker p)^2 = 0$. Note that there exists a unique structure of B -Module on $I = \ker p$ such that $p(x)y = xy$, for sections x and y of E and I , respectively. The above

diagram is said to be an A -extension of B by a B -Module I ([I], III, 1).

A morphism of extensions of X -Algebras is defined by the following commutative diagram of X -Algebras:

$$\begin{array}{ccc} E' & \longrightarrow & E \\ \uparrow \searrow & & \uparrow \searrow \\ & B' \longrightarrow & B \\ \uparrow \nearrow & & \uparrow \nearrow \\ A' & \longrightarrow & A \end{array}$$

Then there exists naturally a homomorphism of B' -Modules:

$$u: I' = \ker p' \longrightarrow I = \ker p$$

where I is regarded as a B' -Module, i. e., a homomorphism of B -Modules

$$u^\# : I' \otimes_{B'} B \longrightarrow I.$$

The set of isomorphism classes of A -extensions of B by a B -Module I forms an abelian group and is denoted by $\text{Exal}_A(B, I)$. The zero element is the class of the trivial extension defined by the ring of dual numbers, $B \oplus I$. The group of automorphisms of an A -extension of B by I coincides with the group of automorphisms of the trivial extension, i. e., the group of A -derivations of B to I , which is denoted by $\text{Der}_A(B, I)$.

1.1.1 Let $0 \rightarrow I \rightarrow E \rightarrow B \rightarrow 0$ be an A -extension. Let $u: C \rightarrow B$ be a morphism of A -Algebras. This induces an A -extension:

$0 \rightarrow I \rightarrow E \times_B C \rightarrow C \rightarrow 0$ ([EGA], 18. 2. 5). This defines a group homomorphism $*u: \text{Exal}_A(B, I) \rightarrow \text{Exal}_A(C, I)$ by $e(E) * u = e(E \times_B C)$ for $e(E) \in \text{Exal}_A(B, I)$.

1.1.2 If $u': C' \rightarrow C$ is a morphism of A -Algebras, one has $e(E) * (u \circ u') = (e(E) * u) u'$ in $\text{Exal}_A(C', I)$.

1.1.3 Given a morphism of A -extensions

$$\begin{array}{ccccccc} 0 & \rightarrow & I & \rightarrow & E & \rightarrow & B \rightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow C \\ 0 & \rightarrow & I' & \rightarrow & E' & \rightarrow & C' \rightarrow 0, \end{array}$$

there exists a unique morphism of A -extensions such that one has the next commutative diagram:

$$\begin{array}{ccccccc} 0 & \rightarrow & I & \rightarrow & E & \rightarrow & B \rightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \rightarrow & I & \rightarrow & E \times_B C & \rightarrow & C \rightarrow 0. \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \rightarrow & I' & \rightarrow & E' & \rightarrow & C' \rightarrow 0 \end{array}$$

1.1.4 Given a homomorphism of B -Modules $v: I \rightarrow J$,

one has an A -extension induced by v

$$0 \rightarrow J \rightarrow E \oplus J \rightarrow B \rightarrow 0; \text{ hence one has a group homomorphism}$$

$$v^*: \text{Exal}_A(B, I) \rightarrow \text{Exal}_A(B, J)$$

by $v^* e(E) = e(E \oplus J)$ for $e(E) \in \text{Exal}_A(B, I)$.

1.1.5 If $v': J \rightarrow J'$ is a homomorphism of B -Modules, one has $(v' \circ v)^* e(E) = v'^* (v^* e''(E))$ in $\text{Exal}_A(B, J')$.

1.1.5.1 Given a morphism of A -extensions

$$\begin{array}{ccccccc}
0 & \rightarrow & J' & \rightarrow & E' & \rightarrow & C \rightarrow 0 \\
& & \uparrow & & \uparrow & & \uparrow \\
& & J & & & & \\
& & \uparrow & & & & \\
0 & \rightarrow & I & \rightarrow & E & \rightarrow & B \rightarrow 0,
\end{array}$$

there exists a unique morphism of A-extensions

$$\begin{array}{ccccccc}
0 & \rightarrow & J' & \rightarrow & E' & \rightarrow & C \rightarrow 0 \\
& & \uparrow & & \uparrow & & \uparrow \\
0 & \rightarrow & J & \rightarrow & E \oplus J & \rightarrow & B \rightarrow 0 \\
& & \uparrow & & \uparrow & & \uparrow \\
0 & \rightarrow & I & \rightarrow & E & \rightarrow & B \rightarrow 0.
\end{array}$$

1.1.6 Given a morphism of A-extensions

$$\begin{array}{ccccccc}
0 & \rightarrow & J & \rightarrow & F & \rightarrow & C \rightarrow 0 \\
& & \uparrow v & & \uparrow w & & \uparrow u \\
0 & \rightarrow & I & \rightarrow & E & \rightarrow & B \rightarrow 0,
\end{array}$$

one has $e(F \times_c B) = e(E \oplus J)$ in $\text{Exal}_A(B, J)$, i. e.,

$$\begin{array}{ccccccc}
0 & \rightarrow & J & \rightarrow & E & \rightarrow & C \rightarrow 0 \\
& & \uparrow & & \uparrow & & \uparrow \\
0 & \rightarrow & J & \rightarrow & E \times_c B & \rightarrow & B \rightarrow 0 \\
& & \uparrow & & \uparrow & & \uparrow \\
0 & \rightarrow & J & \rightarrow & E \oplus J & \rightarrow & B \rightarrow 0 \\
& & \uparrow & & \uparrow & & \uparrow \\
0 & \rightarrow & I & \rightarrow & E & \rightarrow & B \rightarrow 0.
\end{array}$$

1.1.7 If w is absent in the first diagram of 1.1.6, the difference $e(F) * u - v * e(E)$ is said to be the obstruction for the prolongation of u and is denoted by $\delta(u) \in \text{Exal}_A(B, J)$. By definition, we see that the obstruction $\delta(u)$ vanishes if and only if there exists a morphism of A-extensions of 1.1.6.

1.1.8 Let $A \rightarrow B$ be a morphism of X-Algebras. Let $P_A(B) \rightarrow B$ be a standard free resolution of B over A ([I], I, 1.5.5.6). One obtains the B-simplicial Module $\Omega_{P/A}^1$ of Kähler differentials of P over A where $P = P_A(B)$ ([I], I, 1.1.4). Note that $\Omega_{P/A}^1$ is P -flat. The B-simplicial Module $\Omega_{P/A}^1 \otimes_P B$ is called the cotangent complex of B over A and is denoted by $L_{B/A}$. Then one has an augmentation $L_{B/A} \rightarrow \Omega_{B/A}^1 \rightarrow O$ and $H_0(L_{B/A}) = \Omega_{B/A}^1$.

1.1.9 Let $A \rightarrow B \rightarrow C$ be morphisms of X-Algebras and M a C-Module. Then one has a distinguished triangle in the derived category of C-Modules $D(C)$:

$$\begin{array}{ccc}
& L_{C/B} & \\
(1) \swarrow & & \searrow \\
L_{B/A} \otimes_B C & \longrightarrow & L_{C/A}
\end{array}$$

Thus one has an exact sequence

$$\begin{aligned}
0 & \rightarrow \text{Der}_B(C, M) \rightarrow \text{Der}_A(C, M) \rightarrow \text{Der}_A(B, M) \\
& \rightarrow \text{Exal}_B(C, M) \rightarrow \text{Exal}_A(C, M) \rightarrow \text{Exal}_A(B, M) \\
& \rightarrow \mathbf{Ext}_C^2(L_{C/B}, M) \rightarrow \dots
\end{aligned}$$

Note that $\text{Der}_A(B, M) \simeq \mathbf{Ext}_B^0(L_{B/A}, M)$ and $\text{Exal}_A(B, M) \simeq \mathbf{Ext}_B^1(L_{B/A}, M)$.

1.1.10 Let $A \rightarrow B$ be an epimorphism of X-Algebras with the kernel I . Then one has $H_0(L_{B/A}) = O$ and $H_1(L_{B/A}) = I/I^2([I])$.

1.1.11 Given a morphism of A-extensions

$$\begin{array}{ccccccc}
0 & \rightarrow & J & \rightarrow & C & \rightarrow & C_0 \rightarrow 0 \\
& & u \uparrow & & f \uparrow & & f_0 \uparrow \\
0 & \rightarrow & I & \rightarrow & B & \rightarrow & B_0 \rightarrow 0 .
\end{array}$$

one has the following properties ;

(i) The induced homomorphism of C_0 -Modules $u^* : I \otimes_{B_0} C_0 \rightarrow J$ is

surjective if and only if the square $\begin{array}{ccc} C & \longrightarrow & C_0 \\ \uparrow & & \uparrow \\ B & \longrightarrow & B_0 \end{array}$ is cocartesian.

(ii) Assume one of the conditions of (i). Then C is B -flat if and only if C_0 is B_0 -flat and u^* is an isomorphism.

1.2.1 Let X be a scheme. Given the following diagram of A -extensions of R -Algebras, A and R being an R -Algebra and X -Algebra, respectively :

$$\begin{array}{ccccccc}
0 & \rightarrow & L & \rightarrow & D & \rightarrow & D_0 \rightarrow 0 \\
& & w \uparrow & & & & c_0 \uparrow \\
0 & \rightarrow & K & \rightarrow & C & \rightarrow & C_0 \rightarrow 0 \\
& & v \uparrow & & & & b_0 \uparrow \\
0 & \rightarrow & J & \rightarrow & B & \rightarrow & B_0 \rightarrow 0 \\
& & u \uparrow & & \uparrow & & a_0 \uparrow \\
0 & \rightarrow & I & \rightarrow & A & \rightarrow & A_0 \rightarrow 0 .
\end{array}$$

where $u : I \rightarrow J$, $v : J \rightarrow K$ and $w : K \rightarrow L$ are A_0 , B_0 and C_0 -homomorphisms, respectively and where $a_0 : A_0 \rightarrow B_0$, $b_0 : B_0 \rightarrow C_0$ and $c_0 : C_0 \rightarrow D_0$ are R -Algebra morphisms, then one has

$$\delta(c_0 \circ b_0) = w^* \delta(b_0) + \delta(c_0) * b_0 \text{ in } \text{Exal}_{A_0}(B_0, L).$$

Proof.

Step(i) By definition, $\delta(b_0) \in \text{Exal}_A(B_0, K)$. But $\delta(b_0) \in \text{Exal}_{A_0}(B_0, K)$.

Indeed, one has an exact sequence derived from $A \rightarrow A_0 \rightarrow B_0$ as in 1.1.9,

$$0 \rightarrow \text{Exal}_{A_0}(B_0, K) \rightarrow \text{Exal}_A(B_0, K) \xrightarrow{*a_0} \text{Exal}_A(A_0, K)$$

and we see that $\delta(b_0) * a_0 = e(C \times_{C_0} B_0) \times_{B_0} A_0 - e(B \oplus^I K) \times_{B_0} A_0$ vanishes.

By 1.1.2, one has $e((C \times_{C_0} B_0) \times_{B_0} A_0) = e(C \times_{C_0} A_0)$ in $\text{Exal}_A(A_0, K)$.

By 1.1.6, one has $e(B \oplus^I K) = e(C \times_{C_0} A_0)$ in $\text{Exal}_A(A_0, K)$.

Considering morphisms of R -extensions :

$$\begin{array}{ccccccc}
0 & \rightarrow & K & \rightarrow & C & \rightarrow & C_0 \rightarrow 0 \\
& & v \circ u \uparrow & & \uparrow & & \uparrow b_0 \circ a_0 \\
0 & \rightarrow & I & \rightarrow & B & \rightarrow & A_0 \rightarrow 0
\end{array}$$

and

$$\begin{array}{ccccccc}
0 & \rightarrow & K & \rightarrow & B \oplus^I K & \rightarrow & B_0 \rightarrow 0 \\
& & \uparrow & & \uparrow & & \uparrow \\
0 & \rightarrow & J & \rightarrow & B & \rightarrow & B_0 \rightarrow 0 \\
& & \uparrow & & \uparrow & & \uparrow \\
0 & \rightarrow & I & \rightarrow & A & \rightarrow & A_0 \rightarrow 0 .
\end{array}$$

one has $e((B \oplus^I K) \times_{B_0} A_0) = e(A \oplus^I K)$ in $\text{Exal}_A(A_0, K)$.

Therefore, $\delta(b_0) * a_0 = 0$ and so $\delta(b_0) \in \text{Exal}_{A_0}(B_0, K)$.

Step(ii) By (i), one has

$$\delta(b_0) = e(C) * b_0 - v^* e(B) \in \text{Exal}_{A_0}(B_0, K)$$

$$\delta(c_0) = e(D)^*c_0 - w^*e(C) \in \text{Exal}_{A_0}(C_0, L)$$

$$\text{and } \delta(c_0 \circ b_0) = e(D)^*(c_0 \circ b_0) - (w \circ v)^*e(B) \in \text{Exal}_{A_0}(B_0, L).$$

$$\text{One has } w^*\delta(b_0) = w^*(e(C)^*b_0) - w^*(v^*e(B)),$$

$$\delta(c_0)^*b_0 = (e(D)^*c_0)^*b_0 - (w^*e(C))^*b_0, \text{ respectively.}$$

Since $w^*(v^*e(B)) = (w \circ v)^*e(B)$ and $(e(D)^*c_0)^*b_0 = e(D)^*(c_0 \circ b_0)$, we have

$$(e(D)^*c_0)^*b_0 - w^*(v^*e(B)) = \delta(c_0 \circ b_0) \text{ and hence it suffices to show that } w^*(e(C)^*b_0) = (w^*e(C))^*b_0.$$

Step(iii) Considering the following two diagrams of A-extensions

$$\begin{array}{ccccccc} 0 & \rightarrow & K & \rightarrow & C & \rightarrow & C_0 \rightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \rightarrow & K & \rightarrow & C \times_{c_0} B_0 & \rightarrow & B_0 \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & L & \rightarrow & (C \times_{c_0} B_0) \oplus {}^K L & \rightarrow & B_0 \rightarrow 0 \end{array}$$

and

$$\begin{array}{ccccccc} 0 & \rightarrow & K & \rightarrow & C & \rightarrow & C_0 \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & L & \rightarrow & C \oplus {}^K L & \rightarrow & C_0 \rightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \rightarrow & L & \rightarrow & (C \oplus {}^K L) \times_{c_0} B_0 & \rightarrow & B_0 \rightarrow 0 \end{array}$$

one has the next morphisms of A-extensions

$$\begin{array}{ccccccc} 0 & \rightarrow & L & \rightarrow & C \oplus {}^K L & \rightarrow & C_0 \rightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \rightarrow & K & \rightarrow & C & \rightarrow & C_0 \rightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \rightarrow & K & \rightarrow & C \times_{c_0} B_0 & \rightarrow & B_0 \rightarrow 0 \end{array}.$$

Hence one obtains from 1.1.6 an A-isomorphism,

$$\begin{array}{ccccccc} 0 & \rightarrow & L & \rightarrow & (C \oplus {}^K L) \times_{c_0} B_0 & \rightarrow & B_0 \rightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \rightarrow & L & \rightarrow & (C \times_{c_0} B_0) \oplus {}^K L & \rightarrow & B_0 \rightarrow 0 \end{array}$$

$$\text{i. e., } w^*(e(c)^*b_0) = (w^*e(c))^*b_0. \quad \square$$

1.2.2 Let be a scheme and X, Y, Z, T S -schemes. Given a diagram of $p^{-1}(\mathcal{O}_T)$ -extensions of $(s \circ p)^{-1}(\mathcal{O}_S)$, $p^{-1}(\mathcal{O}_T)$ and $(s \circ p)^{-1}(\mathcal{O}_S)$ being $(s \circ p)^{-1}(\mathcal{O}_S)$ and X -Algebras, respectively :

$$\begin{array}{ccccccc} 0 & \rightarrow & L & \rightarrow & \mathcal{O}_X & \rightarrow & \mathcal{O}_{X_0} \rightarrow 0 \\ & & \uparrow & & & & \uparrow \\ 0 & \rightarrow & f_0^{-1}(K) & \rightarrow & f_0^{-1}(\mathcal{O}_Y) & \rightarrow & f_0^{-1}(\mathcal{O}_{Y_0}) \rightarrow 0 \\ & & \uparrow & & & & \uparrow \\ 0 & \rightarrow & h_0^{-1}(J) & \rightarrow & h_0^{-1}(\mathcal{O}_Z) & \rightarrow & h_0^{-1}(\mathcal{O}_{Z_0}) \rightarrow 0 \\ & & \uparrow & & & & \uparrow \\ 0 & \rightarrow & h_0^{-1}(I) & \rightarrow & p_0^{-1}(\mathcal{O}_T) & \rightarrow & p_0^{-1}(\mathcal{O}_{T_0}) \rightarrow 0 \\ & & & & \uparrow & & \\ & & & & (s_0 \circ p_0)^{-1}(\mathcal{O}_S) & & \end{array}$$

where T is an S -extension by I and where X, Y, Z are T -extensions by L, K, J , respectively, and where there exist the following commutative diagrams

$$\begin{array}{ccc}
 & X_0 & \\
 f_0 \swarrow & & \searrow h_0 = g_0 \circ f_0 \\
 Y_0 & \xrightarrow{g_0} & Z_0 \\
 q_0 \searrow & & \swarrow r_0 \\
 & T_0 & \\
 & s_0 \searrow & \\
 & S &
 \end{array}
 \qquad
 \begin{array}{ccc}
 & X & \\
 & \downarrow p & \\
 Y & & Z \\
 q \searrow & & \swarrow r \\
 & T & \\
 & \swarrow s & \\
 & S &
 \end{array}$$

one has a relation of obstructions of prolongation

$$\delta(h_0) = w^* \delta(g_0) + \delta(f_0)^* g_0 \quad \text{in} \quad \mathbf{Ext}^1(X_0; h_0^* L_{Z_0/T_0}, L).$$

This is an interpretation of 1.2.1.

Note that corresponding to the following three morphisms of \mathcal{O}_{X_0} -modules

$$\begin{aligned}
 w^* &: f_0^*(K) \rightarrow L \\
 v^* &: h_0^*(J) \rightarrow f_0^*(K), \\
 u^* &: p_0^*(I) \rightarrow h_0^*(J)
 \end{aligned}$$

one has

$$\mathbf{Ext}^1(Y_0; g_0^*(L_{Z_0/T_0}), K) \xrightarrow{w^*} \mathbf{Ext}^1(X_0; h_0^*(L_{Z_0/T_0}), L)$$

and

$$\mathbf{Ext}^1(X_0; f_0^*(L_{Y_0/T_0}), L) \xrightarrow{*g_0} \mathbf{Ext}^1(X_0; h_0^*(L_{Z_0/T_0}), L)$$

1.2.3 Let X, Y be irreducible reduced schemes and $f: X \rightarrow Y$ a finite separable (not necessarily flat) surjective morphism. Assume that Y is normal. Then one has a trace homomorphism

$$\mathrm{tr}: f_* \mathcal{O}_X \longrightarrow \mathcal{O}_Y.$$

Note that the composite with $\rho: \mathcal{O}_Y \rightarrow f_* \mathcal{O}_X$

$$\mathrm{tr} \circ \rho: \mathcal{O}_Y \longrightarrow \mathcal{O}_Y$$

is multiplication by $\deg f$ unless the characteristic of Y divides $\deg f$. For a convenience to the reader, we recall the definition of the trace map: let U be an affine open set of Y . We have a commutative

$$\begin{array}{ccc}
 \Gamma(f^{-1}(U), \mathcal{O}_X) & \subset & R(X) \\
 \uparrow & & \uparrow \\
 \Gamma(U, \mathcal{O}_Y) & \subset & R(Y)
 \end{array}$$

square

Thus one defines a homomorphism

$$\mathrm{tr}(U): \Gamma(f^{-1}(U), \mathcal{O}_X) \longrightarrow \Gamma(U, \mathcal{O}_Y)$$

which maps a section x to the sum of the conjugates x^τ where the τ run over all extensions of the identity of $R(Y)$ to the Galois closure of $R(X)/R(Y)$.

1.2.3.1 Even if f is a proper separable surjective morphism, we can define $\mathrm{tr}: f_* \mathcal{O}_X \rightarrow \mathcal{O}_Y$, taking Stein factorization of f .

1.2.4 Let X, Y be irreducible reduced algebraic k -schemes and $f: X \rightarrow Y$ a proper separable surjective morphism. Assume that Y is normal.

Let E be an object of the derived category $D^-(Y)$ of the complexes bounded above and F a flat \mathcal{O}_Y -Module.

Corresponding to the homomorphism $F \rightarrow f_* f^* F \rightarrow Rf_* f^* F$, one has $\mathrm{Rhom}(E, F) \rightarrow \mathrm{Rhom}(E, f_* f^* F) \rightarrow \mathrm{Rhom}(E, Rf_* f^* F)$. Since we have a spectral sequence

$$E_2^{p,q} = H^p(\mathrm{RHom}(E, R^q f_* f^* F)) = > E^{p+q} = H^{p+q}(\mathrm{Rhom}(E, Rf_* f^* F)) =$$

$\mathbf{Ext}^{p+q}(X; Lf^* E, f^* F) = \mathbf{Ext}^{p+q}(X; f^* E, f^* F)$, there exists exact sequence

$$0 \rightarrow E_2^{1,0} = \mathbf{Ext}^1(Y; E, f_* f^* F) \rightarrow E^1 = \mathbf{Ext}^1(X; f^* E, f^* F).$$

By 1.2.3.1, one has also an injection $\mathbf{Ext}^1(Y; E, F) \rightarrow \mathbf{Ext}^1(Y; E, f_* \mathcal{O}_X \otimes F)$, unless the characteristic of k divides $\deg g$ where g is the finite morphism derived from Stein factorization of f .

1.2.5 Under the hypothesis of 1.2.2, suppose moreover that

- (i) $J = r_0^*(I)$, $K = q_0^*(I)$ and $L = p_0^*(I)$
- (ii) p_0, q_0, r_0 are flat morphisms
- (iii) I is \mathcal{O}_{T_0} -flat
- (iv) X_0, Y_0 are irreducible reduced algebraic k -scheme
- (v) $f_0: X_0 \rightarrow Y_0$ is a proper separable surjective morphism
- (vi) Y_0 is normal
- (vii) X_0, Y_0, Z_0 are the pullbacks of X, Y, Z along $T_0 \rightarrow T$
- (viii) the characteristic of k does not divide the degree of the finite morphism of Stein factorization f_0
- (ix) there exist T -morphisms $f: X \rightarrow Y$ and $h: X \rightarrow Z$.

Then there exists a T -morphism $g: Y \rightarrow Z$ and moreover p, q, r are flat morphisms.

Proof. By 1.2.2, 1.2.3.1, 1.2.4 and 1.1.11, one can easily prove this, because $\delta(f_0) = \delta(h_0) = 0$ implies $\delta(g_0) = 0$.

1.2.6 Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms of schemes. Then one has the distinguished triangle corresponding to $(g \circ f)^{-1}(\mathcal{O}_Z) \rightarrow f^{-1}(\mathcal{O}_Y) \rightarrow \mathcal{O}_X$,

$$\begin{array}{ccc} & L_{X/Y} & \\ (1) \swarrow & & \nwarrow \\ f^* L_{Y/Z} & \rightarrow & L_{X/Y} \end{array} \quad \text{in the derived category } D(X).$$

The degree one morphism of the diagram above is called the class of Kodaira-Spencer of the

diagram $\begin{array}{ccc} X & \longrightarrow & Y \\ & \searrow & \swarrow \\ & Z & \end{array}$ and is denoted by $K(X/Y/Z)$.

Note that it is an element of $\mathrm{Hom}_{D(X)}(L_{X/Y}, f^* L_{Y/Z}[1]) = \mathrm{Ext}^1(L_{X/Y}, f^* L_{Y/Z})$ ([I], II, 2.1.5).

1.2.7 Suppose that $g: Y \rightarrow Z$ is smooth. Then $L_{Y/Z}$ is quasi-isomorphic to $\Omega^1_{Y/Z}$ ([I], III prop. 3.1.2). Let $P^1_{Y/Z}$ be a sheaf of principal parts of order one of a Z -scheme Y (EGA IV 16.3). The ringed space $(Y, P^1_{Y/Z})$ is called the first infinitesimal neighbourhood of Y , which is denoted by $\Delta^1_{Y/Z}$. There exist two natural projections $q_1, q_2: \Delta^1_{Y/Z} \rightarrow Y$. We consider q_1 as the structure morphism of $\Delta^1_{Y/Z}$ over Y .

1.2.7.1 The cartesian pullback X' of a Y -scheme X by $q_1: \Delta^1_{Y/Z} \rightarrow Y$ and the cartesian pullback X'' of the Y -scheme X by $q_2: \Delta^1_{Y/Z} \rightarrow Y$ determine $e(X'') - e(X') \in \mathbf{Ext}^1(X; L_{X/Y}, f^* \Omega^1_{Y/Z})$ (1.1.6, 1.1.10). Since $\Delta^1_{Y/Z}$ is regarded as a Y -scheme by p_1 , $e(X')$ is zero in $\mathbf{Ext}^1(X, L_{X/Y}, f^* \Omega^1_{X/Y})$ and X'' can be regarded as a Y -extension of X by $f^* \Omega^1_{Y/Z}$.

1.2.7.2 Let $\Delta^1_{Y/Z, y}$ be the pullback of the Y -scheme $\Delta^1_{Y/Z}$ on a point y of Y , X''_y the pullback of the $\Delta^1_{Y/Z}$ -scheme X'' by the canonical inclusion $\Delta^1_{Y/Z, y} \rightarrow \Delta^1_{Y/Z}$. Considering the following cartesian

diagram

$$\begin{array}{ccccc}
 & X_Y & \rightarrow & X & \\
 & \swarrow & \downarrow & \swarrow & \\
 X''_Y & \rightarrow & X'' & \downarrow & \\
 \downarrow & & \downarrow & & \\
 & y & \rightarrow & Y & \\
 \swarrow & & \swarrow & & \\
 \Delta^1_{Y/Z, Y} & \rightarrow & \Delta^1_{Y/Z} & &
 \end{array}$$

one has

$$\begin{array}{ccc}
 e(X'') & \in & \mathbf{Ext}^1(X; L_{X/Y}, f^* \Omega^1_{Y/Z}), \\
 \downarrow & & \downarrow \\
 i^* e(X'') = e(X''_Y) * i & \in & \mathbf{Ext}^1(X_Y; i^* L_{X/Y}, i^* f^* \Omega^1_{Y/Z}), \\
 \uparrow & & \uparrow \\
 e(X''_Y) & \in & \mathbf{Ext}^1(X_Y; L_{X_Y/Y}, f^* j^* \Omega^1_{Y/Z}) \quad \text{by (1.1.6)}.
 \end{array}$$

1.2.8 Let S be a scheme, X, Y S -schemes and $Z = X \times_S Y$; We name the structure morphisms and projections as follows:

$$\begin{array}{ccc}
 & Z & \\
 p \swarrow & & \searrow q \\
 X & & Y \\
 \searrow f & & \swarrow g \\
 & S &
 \end{array}$$

Assume that $\mathrm{Tor}_i^{h^{-1}(\mathcal{O}_S)}(p^{-1}(\mathcal{O}_X), q^{-1}(\mathcal{O}_Y)) = 0$, where $h = f \circ p = g \circ q$, for $i > 0$. Then one has quasi-isomorphisms

$$p^* L_{X/Y} \simeq L_{Z/Y}, \quad p^* L_{X/S} \oplus q^* L_{Y/S} \simeq L_{Z/S}.$$

1.2.9 Let S be a scheme $\mathrm{Spec} k$, T a smooth S -scheme and X, Y, Z proper flat T -schemes in the following way:

$$\begin{array}{ccccc}
 & X & & & \\
 f \swarrow & & \searrow h & & \\
 Y & \xrightarrow{p} & Z & & \\
 q \searrow & & \swarrow r & & \\
 & T & & & \\
 & \downarrow & & & \\
 & S & & &
 \end{array}$$

For a fixed point of T , suppose the following

- (i) X_t, Y_t, Z_t are irreducible and reduced,
- (ii) Y_t, Z_t are normal,
- (iii) there exists a proper surjective morphism $g_t: Y_t \rightarrow Z_t$ such that $h_t = g_t \circ f_t$ and that f_t and g_t are separable,
- (iv) f and h are surjective,
- (v) Y is isomorphic to $Y_t \times T$,
- (vi) the characteristic of k does not divide the degree of a finite part of Stein factorization of f_t and g_t .

Then Kodaira-Spencer class $K(Z/T/S)_t = e(Z''_t)$ vanishes in

$$\mathbf{Ext}^1(Z_t; L_{Z_t/t}, r_t^*(\Omega^1_{T/S} \otimes k(t))).$$

Proof. By 1.2.7.2, one has the following commutative diagrams:

$$\begin{array}{ccc}
& X''_t & \\
f''_t \swarrow & & \searrow h''_t \\
Y''_t & \xrightarrow{\quad} & Z''_t \\
& \Delta^1_{T/S, t} &
\end{array}
\quad
\begin{array}{ccc}
& X_t & \\
f_t \swarrow & & \searrow h_t \\
Y_t & \xrightarrow{g_t} & Z_t \\
q_t \swarrow & & \searrow r_t \\
& t &
\end{array}$$

Applying 1.2.5, one obtains a $\Delta^1_{T/S, t}$ -morphism g''_t of a prolongation of g_t . Hence one has the correspondences by 1.1.6, that is

$$g_t^* e(Z''_t) = e(Y''_t) * g_t, \text{ i.e.,}$$

$$\begin{array}{ccc}
e(Z''_t) & \in & \mathbf{Ext}^1(Z_t; L_{Z_t/t}, r_t^*(\Omega^1_{T/S} \otimes k(t))) \\
\downarrow & & \downarrow \\
g_t^* e(Z''_t) = e(Y''_t) * g_t & \in & \mathbf{Ext}^1(Y_t; g_t^* L_{Z_t/t}, q_t^*(\Omega^1_{T/S} \otimes k(t))) \\
\uparrow & & \uparrow \\
e(Y''_t) & \in & \mathbf{Ext}^1(Y_t; L_{Y_t/t}, q_t^*(\Omega^1_{T/S} \otimes k(t))).
\end{array}$$

By 1.2.4, the above map g_t^* is injective. Since $e(Y''_t)$ is zero by assumption(v), $e(Z''_t)$ is also zero. \square

1.2.9.1

Let X, Y be irreducible reduced algebraic k -schemes and $f: X \rightarrow Y$ a separable dominant morphism. Suppose that Y be smooth. Then the sequence $0 \rightarrow f^* \Omega^1_{Y/k} \rightarrow \Omega^1_{X/k} \rightarrow \Omega^1_{X/Y} \rightarrow 0$ is exact.

Proof. It suffices to show the injectivity of the sequence above.

Let x, y be generic points of X, Y , respectively. Considering the commutative diagram

$$\begin{array}{ccccc}
X & \xleftarrow{j'} & X_y & \xleftarrow{i} & x \\
f \downarrow & & f' \downarrow & & \\
Y & \xleftarrow{j} & y & &
\end{array}
\quad h = j' \circ i$$

one obtains the following commutative diagram :

$$\begin{array}{ccccccc}
0 & \rightarrow & (f' \circ i)^* \Omega^1_{Y/k} & \rightarrow & \Omega^1_{X/k} & \rightarrow & \Omega^1_{X/Y} \rightarrow 0 \\
& & u \uparrow & & v \uparrow & & \\
& & f^* \Omega^1_{Y/k} & \rightarrow & h^* \Omega^1_{X/k} & &
\end{array} \quad (1.2.9.1.1)$$

and u, v are isomorphisms, since j, h are formally étale.

The horizontal sequence (1.2.9.1.1) is exact because of separability of f . By adjoint, the following diagram is commutative

$$\begin{array}{ccc}
0 & \rightarrow & h_* h^* \Omega^1_{Y/k} \rightarrow h_* \Omega^1_{X/k} \\
& & \sigma \uparrow \quad \uparrow \\
& & f^* \Omega^1_{Y/k} \rightarrow \Omega^1_{X/k}
\end{array} \quad (1.2.9.1.2)$$

Since $\Omega^1_{X/Y}$ is flat, σ is injective. Since h is affine, the horizontal sequence (1.2.9.1.2) is exact. This completes the proof. \square

1.2.9.2 Let $X \rightarrow Y \rightarrow Z$ be morphisms of irreducible reduced schemes, $f: X \rightarrow Y$ a dominant separable morphism, $g: Y \rightarrow Z$ a smooth morphism and $g \circ f: X \rightarrow Z$ a flat morphism. Then the sequence

$$0 \rightarrow f^* \Omega^1_{Y/Z} \rightarrow \Omega^1_{X/Z} \rightarrow \Omega^1_{X/Y} \rightarrow 0$$

is exact.

Proof. Let z be generic point of Z . By 1.2.9.2, the homomorphism

$$f_z^* \Omega^1_{Y_z/Z} \rightarrow \Omega^1_{X_z/Z}$$

is injective. The square

$$\begin{array}{ccc} f^* \Omega^1_{Y/Z} & \longrightarrow & \Omega^1_{X/Z} \\ \downarrow & & \downarrow \\ f^* \Omega^1_{Y/Z} \otimes_{\mathcal{O}_Z} \mathcal{O}_{Z,z} & \longrightarrow & \Omega^1_{X_z/Z} \end{array}$$

is commutative.

The left vertical arrow of the diagram above is injective.

1.3 Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms. Suppose that $g: Y \rightarrow Z$ be a smooth morphism, $g \circ f: X \rightarrow Z$ a flat morphism, f, g morphisms of finite type and Z a locally noetherian scheme. One has therefore a morphism of distinguished triangles in $D(X)^-$:

$$\begin{array}{ccccc} & L_{X/Y} & \longrightarrow & \Omega^1_{X/Y} & \\ (1) \swarrow & & & & \swarrow (1) \\ f^* L_{Y/Z} & \longrightarrow & L_{X/Z} & \longrightarrow & \Omega^1_{X/Z} \\ & \nearrow & \nearrow & & \nearrow \\ & f^* \Omega^1_{Y/Z} & \longrightarrow & \Omega^1_{X/Z} & \end{array}$$

by 1.2.9.3.

Since g is smooth, one has an isomorphism

$$\mathrm{Hom}_{D(X)}(L_{X/Y}, f^* \Omega^1_{Y/Z}[1]) \simeq \mathrm{Hom}_{D(X)}(L_{X/Y}, f^* L_{Y/Z}[1]) \text{ and we have an inclusion } \\ \mathrm{Hom}_{D(X)}(\Omega^1_{X/Y}, f^* \Omega^1_{Y/Z}[1]) \subset \mathrm{Hom}_{D(X)}(L_{X/Y}, f^* \Omega^1_{Y/Z}[1]).$$

The slant degree one arrow of the right triangle of the diagram above is denoted by $c(X/Y/Z)$, which corresponds to Kodaira-Spencer class $K(X/Y/Z)$ by the above inclusion.

1.3.1 We let $\Theta_{X/Y}$ denote $\mathrm{Hom}(\Omega^1_{X/Y}, \mathcal{O}_X)$. Under the assumption of 1.3, assume furthermore that $c(X/Y/Z)$ vanishes. Then the following sequence is exact:

$$0 \rightarrow \Theta_{X/Y} \rightarrow \Theta_{X/Z} \rightarrow f^* \Theta_{Y/Z} \rightarrow 0.$$

Proof. It suffices to show that the connecting homomorphism

$$d: f^* \Theta_{Y/Z} \rightarrow \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_{X/Y}, \mathcal{O}_X)$$

is a zero map. The map d corresponds to the image of $c(X/Y/Z)$ via the following map which is denoted by $d(X/Y/Z)$

$$H^1(R\Gamma(X, R\mathrm{Hom}(\Omega^1_{X/Y}, f^* \Omega^1_{Y/Z}))) \rightarrow \Gamma(X, H^1(R\mathrm{Hom}(\Omega^1_{X/Y}, f^* \Omega^1_{Y/Z}))).$$

$$\begin{aligned} \text{Here } H^1(R\mathrm{Hom}(\Omega^1_{X/Y}, f^* \Omega^1_{Y/Z})) &= H^1(R\mathrm{Hom}(\Omega^1_{X/Y}, \mathcal{O}_X) \otimes f^* \Omega^1_{Y/Z}) \\ &= \mathrm{Ext}^1(\Omega^1_{X/Y}, \mathcal{O}_X) \otimes f^* \Omega^1_{Y/Z}. \end{aligned}$$

We use the following lemma to prove the first isomorphism.

Lemma. Let X be a scheme. Then there exists a natural functorial homomorphism $R\mathrm{Hom}(F, G) \xrightarrow{L} H \rightarrow R\mathrm{Hom}(F, G \otimes^L H)$

for $F \in D^-(X)$, $G \in D^+(X)$ and $H \in D^b(X)_{\mathrm{fTd}}$, where $D^b(X)_{\mathrm{fTd}}$ means the derived category of bounded complexes of finite Tor-dimension.

If furthermore X is locally noetherian and $F \in D^-_c(X)$ (=the derived category of bounded above complexes of coherent Modules), then it is an isomorphism ([H], Prop. 5.14).

1.3.2 Under the assumption of 1.3, assume furthermore that $d(X/Y/Z)$ vanishes. Then $c(X/Y/Z) \in H^1(X, \Theta_{X/Y} \otimes f^* \Omega^1_{Y/Z})$.

Indeed, there exists an exact sequence of Edge-homomorphisms

$$\begin{array}{c}
 0 \\
 \searrow \\
 H^1(X, \text{Hom}(\Omega_{X/Y}^1, f^*\Omega_{Y/Z}^1)) \\
 \searrow \\
 c(X/Y/Z) \in H^1(R\text{Hom}((\Omega_{X/Y}^1, f^*\Omega_{Y/Z}^1))) \\
 \searrow \quad \searrow \\
 0 = d(X/Y/Z) \in \Gamma(X, f^*\Omega_{Y/Z}^1 \otimes \text{Ext}^1(\Omega_{Y/Z}^1, \mathcal{O}_X))
 \end{array}$$

Hence $c(X/Y/X) \in H^1(X, \text{Hom}(\Omega_{X/Y}^1, f^*\Omega_{Y/Z}^1)) \simeq H^1(R\Gamma(Y, Rf_*(\Theta_{X/Y} \otimes f^*\Omega_{Y/Z}^1)))$.

The image of $c(X/Y/Z)$ by the next map is denoted by $\rho(X/Y/Z) : H^1(R\Gamma(Y, Rf_*(\Theta_{X/Y} \otimes f^*\Omega_{Y/Z}^1))) \rightarrow \Gamma(Y, R^1f_*\Theta_{X/Y} \otimes \Omega_{Y/Z}^1) \simeq \text{Hom}(\Theta_{Y/Z}, R^1f_*\Theta_{X/Y})$.

1.3.2.1 Let F, G be coherent sheaves on a scheme X of finite type over an irreducible reduced affine scheme S . For any integer q , there exists a non-empty open set U of S such that for each point s of U the canonical map $\text{Ext}_{\mathcal{O}_X}^q(F, G)_s \rightarrow \text{Ext}_{\mathcal{O}_{X_s}}^q(F_s, G_s)$

is an isomorphism, where the subscript s denotes the restriction to the fibre over s ([A], Lemma 6.8).

1.3.2.2 Let $f : X \rightarrow Y$ be a proper S -morphism, where S is an irreducible reduced noetherian affine scheme let F be a coherent sheaf on X . For each integer q , there exists a non-empty open set U of S such that for any point s of U the canonical map $(R^q f_* F)_s \rightarrow R^q f_{s*} F_s$ is an isomorphism ([A]).

1.3.2.3 Let A be a reduced Jacobson ring, $f : M \rightarrow N$ an A -homomorphism, where M, N are free modules. If for each closed point x of $\text{Spec } A$ $f_x : M \otimes k(x) \rightarrow N \otimes k(x)$ is a zero map, then so is f . Proof. Representing f by a matrix, each element of the matrix is contained in the Jacobson radical. \square

1.3.2.4 Under the assumptions except (v) of 1.2.9, assume further that

- (i) $S = \text{Spec } k$ and $K(Z/T/S)_s = 0$ for any closed point $s \in T$,
- (ii) T is of finite type over k and X, Y, Z are connected over T . Then there exists an open dense affine subscheme U of T such that

$$\Gamma(Z_U, \Theta_{Z_U/k}) \rightarrow \Gamma(U, \Theta_{U/k}) \rightarrow 0$$

is exact.

Proof. It is a direct consequence of 1.2.9.2, 1.3.1, 1.3.2, 1.3.2.1, 1.3.2.2 and 1.3.2.3.

1.4 Definition. Let X be a complex space, X' a relatively compact subset of X . A one-parameter group of transformations in X' means a

$$\Phi : \{s \in \mathbb{C} : |s| < \varepsilon\} \times X' \rightarrow X, \quad \varepsilon > 0 \text{ such that } \Phi(s_1, \Phi(s_2, x)) =$$

$$\Phi(s_1 + s_2, x) \text{ for } |s_1|, |s_2|, |s_1 + s_2| < \varepsilon \text{ and } \Phi(0, x) = \text{id} \text{ and that for a fixed } s \text{ the mapping } \Phi : \{s\} \times X' \rightarrow \Phi(s, X') \text{ is biholomorphic ([G])}.$$

1.4.1 **Lemma** (Grauert).

Let (X, \mathcal{O}_X) be a complex space and $v \in \Gamma(X, \Theta_X)$ a holomorphic vector field on X . Then there exists a one-parameter group of transformations Φ in every relatively compact subset X' of X such that for $f \in \mathcal{O}_{X, x}$, $x \in X'$

$$\lim_{s \rightarrow 0} \frac{f(\Phi(s, x)) - f(x)}{s} = v_x(df)$$

([G], Hilfsatz 3.2).

Theorem (Grauert). Let $\pi: X \rightarrow M$ be a proper flat and reduced surjective morphism from X onto a complex manifold M . If for $x \in X$ the sequence $\Theta_{x,x} \rightarrow \pi^* \Theta_{M,x} \rightarrow 0$ is exact, then $\pi: X \rightarrow M$ is locally trivial at $x \in X$ ([G], Satz 3.3).

1.5 Thus we obtain the following theorem.

Theorem. Suppose everything is defined over the complex number field. Let T be a manifold, X, Y, Z irreducible reduced T -schemes of finite type and $p: X \rightarrow T, q: Y \rightarrow T, r: Z \rightarrow T$ flat morphisms over T . Assume the next commutative diagram

$$\begin{array}{ccccc} & & X & & \\ f \swarrow & & & \searrow h & \\ Y & & p \downarrow & & Z \\ q \searrow & & & \swarrow r & \\ & & T & & \end{array}$$

such that

- (i) f, h are proper surjective morphisms
 - (ii) Y is biholomorphically trivial over T , i. e., $\simeq Y_0 \times T$ for some Y_0 .
 - (iii) there exists proper surjective morphism $g_t: Y_t \rightarrow Z_t$ with $h_t = g_t \circ f_t$.
- Then Z is biholomorphically trivial over a non-empty open subset T^0 of T .

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