Remarks on Esnault-Viehweg's Results

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Introduction

After Tankeev-Kollăr vanishing theorem appeared, several kinds of such theorems are proved. With classification of open algebraic varieties [I] in mind, we want to prove a vanishing theorem for logarithmic differentials. Actually, our theorem results if we combine Deligne-Illusie ([DI], [A 1]) M. H. S. on relative cohomology groups for open algebraic varieties with the latter half of Logarithmic De Rham complexes and vanithing theorems of Esnault-Viehweg's ([EV3]). The D -module theoretic notion of Hodge Module constructed by M. Saito [Sa] seems to be useful to generalize the above results. Of course, he did it but I hope there would be more. On any dimensional varieties Hodge theory with degenerating coefficients has been constructed by Y. Shimizu [Si]. This is a generalization of Zucker's theory. We appreciate Professors Esnault-Viehweg for their comment, which points out to the author that in order to show Theorem 1, except for the idea of some device for the good covering, the same argument works well as in the proof of [EV] (see Remark 1) without [DI], [A1].

Notation. D is called a Q-divisor if D is a member of Div $(X)\otimes Q$.

We let $[D]$, $\{D\}$, and $\langle D\rangle$ denote the integral, fractional parts and the round up, respectively, i, e., $\{D\}=D-[D]$, $\langle D\rangle=-[-D]$. If L is an invertible sheaf and $L^N=O(D)$, then $O(\{\frac{j}{N}D\})$ is defined by $L^j(-[\frac{j}{N}D])$. Let (X, O_X) be a ringed space, $\mathscr{M}_X$ the sheaf of germs of meromorphic functions, $\mathscr{M}_X{}^*$ the sheaf of germs of regular meromorphic functions and $O_X{}^*$ the sheaf of germs of invertible sections of O_X. We denote $\mathscr{M}_X{}^*/O_X{}^*$ by Div_X. A divisor means an element of $H^0(X, Div_X)=\mathrm{Div}(X)$ Letting L be an invertible sheaf on X, we denote ay $\mathscr{M}_X(L)^*$ the subsheaf of $\mathscr{M}_X(L)$ such that for any open set U, $\Gamma(U, \mathscr{M}_X(L)^*)$ consists of all the regular meromorphic sections of L over U. For a local system V, let V^v denote $Hom_{C_X}(V, C_X)$ where $\boldsymbol{C}_X$ is a simple sheaf with a typical stalk $\boldsymbol{C}$. We denote $H^q(X, \Omega_X^p(\log D))$ by $H^q(\Omega_X^p\langle D\rangle)$ in short.

§1.

The following theorem is obtained by combining Esnault-Viehweg's Theorem [EV] in case $E=0$ and Deligne-Illusie's ([DI], [A1]) of Mixed Hodge structure on the relative cohomology.

Theorem 1. *Let X be a compact complex manifold which in bimeromorphically dominated by a Kähler manifold. Let L be an invertible sheaf of X. Let E and D be* effective *divisors on X with no common component such that E in a reduced divisor* and *that $E+D$ is supported io a normal crossing divisor.*

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Suppose that $L^N = O(D)$ *for some* $N > 1$. *Let* $1 \leq j \leq N-1$.

1) *Let B be an effective divisor supported in* supp $(\{\frac{j}{N}D\})$.

Let ϕ_B be *the map* $H^q(X, O(-B-E)\otimes O(-\{\frac{j}{N}\})) \to H^q(X, O(-E)\otimes O(-\{\frac{j}{N}D\}))$ *induced from the short exact sequence* $0 \to O(-B) \to O \to O_B \to 0$.

Then the maps $\phi_B : H^q(X, O(-B-E)\otimes O(-\{\frac{j}{N}D\})) \to H^q(X, O(-E)\otimes O(-\{\frac{j}{N}D\}))$ *are surjective for all* $p, q \geq 0$.

Therefore, for a Q-divisor $\frac{j}{N}D$, *the maps* ${}^t\phi_B : H^q(X, O(K+E+\{\frac{j}{N}D\})) \to H^q(X, O(K+E+B+\{\frac{j}{N}D\}))$ *are injective by duality.*

2) *Let C be* an *irreducible component of* supp $(\{\frac{j}{N}D\})$ *and* $D' = D_{red} - C$, *Let* ϕ_C *be the map*

$$H^q(\Omega_X^p\langle D'+C+E\rangle(-E-C-\{\frac{j}{N}D\})) \to H^q(\Omega_X^p\langle D'+E\rangle(-E-\{\frac{j}{N}D\}))$$

jnduced from Kodaira-Spencer sequence

$$0 \to \Omega_X^p\langle D'+C+E\rangle(-C) \to \Omega_X^p\langle D'+E) \to \Omega_C^p\langle D'+E\rangle_C \to 0.$$

Then

$$\phi_C : H^q(\Omega_X^p\langle D'+C+E\rangle(-E-C-\{\frac{j}{N}D\})) \to H^q(\Omega_X^p\langle D'+E\rangle(-E-\{\frac{j}{N}D\}))$$

are surjective for all $p, q \geq 0$.

Dually, one obtains that the maps

$${}^t\phi_C : H^q(\Omega_X^p\langle D'+E\rangle(-D'+\{\frac{j}{N}D\})) \to H^q(\Omega_X^p\langle D'+E+C\rangle(-D'+\{\frac{j}{N}D\}))$$

are injective for all $p, q \geq 0$.

Proof. We use the results from theorems of Deligne-Illusie and Esnault-Viehweg ([DI], [A1], [EV3]).

Now we state the following theorem by Deligne-Illusie (Arapura [A1]) :

Lemma 2. *Let X be a compact Kähler manifold and let E and D be effective divisors on X such that E+D in supported on a normal crossing divisor with no common component and* that *E is* reduced. *Let* $X-D := X - D_{red}$.

Then $H^{p+q}(X-D, (X-D)\cap E; Q)$ *carries a mixed Hodge structure with Hodge filtation F and weight filtration W such that*

$$Gr_F^p H^{p+q}(X-D, (X-D)\cap E; C) = H^q(X, \Omega_X^p\langle E+D\rangle(-E)).$$

In particular, the Hodge spectral sequence degenerates :

$$E_1^{p,q} = H^q(\Omega_X^p\langle E+D\rangle(-E)) = H^{p+q}(X, \Omega_X\langle E+D\rangle(-E)).$$

To apply Esnault-Viehweg argument [EV3] to the situation above, we write the exact sequence, which is well-known as Kodaira-Spencer sequence :

$$0 \to \Omega_X^p\langle D+E\rangle(-E-C) \to \Omega_X^p\langle D'+E\rangle(-E) \to \Omega_C^p\langle(D'+E)_C\rangle(-E_C) \to 0,$$

The cohomology sequence above corresponds to an exact sequence of relative cohomology groups of separated topological spaces :

$$\cdots\cdot H^q(X, X') \to H^q(X, X'') \to H^q(X', X'') \to \cdots\cdot$$

for $X \supset X' \supset X''$, where $X := X - D'$, $X' := (X - D') \cap (C + E)$,
$X'' := (X + D') \cap E = E - D' = (C - D') \cap E$.

Following Esnault-Viehweg [EV3, 169], we describe the logarithmic connection of cyclic covering: Let $L^N = O_X(D)$. D induce an O_X-algebra $\bigoplus_{j=0}^{N-1} L^{-j}$. Let Y be a normalization of *Spec* $\bigoplus_{j=0}^{N-1} L^{-j}$. Then $\pi: Y \to X$ is a cyclic cover ramified along the normal crossing divisor D. Let $\sigma: Z \to Y$ be a desingularization of Y such thet $\sigma^{-1}\pi^{-1}(D+E) = \Delta + \Xi$ is a normal crossing divisor, too. Y has rational singularities and $\pi_* O_Y$ is locally free. The triple $((Z, \Delta+\Xi), Y, (X, D+E))$ being a good covering, one has ([EV1, 242]) $\boldsymbol{R}(\pi\circ\sigma)_*\Omega_Z^p\langle\Delta+\Xi\rangle = \Omega_X^p\langle D+E\rangle \otimes_{O_X} \pi_* O_Y$. The Kähler differential $d: O_Z \to \Omega_Z{}^1\langle\Delta + E\rangle$ induces a logarithmic connection ∇ on $\pi_* O_Y$ and one har

$$\boldsymbol{R}(\pi\circ\sigma)_* DR_{\Delta+\Xi} O_Z = \boldsymbol{R}(\pi\circ\sigma)_*\Omega_Z\langle\Delta+\Xi\rangle = DR_{D+E}\pi_* O_Y = \Omega_X\langle D+E\rangle(\pi_* O_Y)$$

LOt $j: Z - (\Delta \cup \Xi) \to Z - \Delta$, $i: Z - \Delta \to Z$, Then from Deligne-Illusie ([DI], [A1, Th2]), denting by $DR_{(Z-\Delta,\Xi)} O_Z = \Omega_Z\langle\Delta+\Xi\rangle(-E)$ the relative logarithmic De Rham complex of the relativ co-homological group and using the isomorphism:

$$H^n(Z-\Delta, (Z-\Delta)\cap\Xi\,;\, Q) = H((Z-\Delta)\,;\, Rj_! Q),$$

one obtains $DR_{(Z-\Delta,\Xi)} O_Z = \boldsymbol{R}i_* \boldsymbol{R}j\boldsymbol{C}_{Z-(\Delta\cup\Xi)} \simeq DR_{\Delta+\Xi} O_Z(-\Xi)$ formally. Using $\boldsymbol{R}(\pi\circ\sigma)_* DR_{\Delta+\Xi}(O_Z(-\Xi)) \simeq DR_{D+E}(\pi_* O_Y(-E))$, one obtains

$$\boldsymbol{R}(\pi\circ\sigma)_* DR_{(Z-\Delta,\Xi)} O_Z \simeq DR_{(X-D,E)}\pi_* O_Y \simeq \Omega_X\langle D+E\rangle(-E)\otimes\pi_* O_Y.$$

Let M be a direct summand of $\pi_* O_Y$, which is invariant under the Galois group G of the covering. The ∇ induces a logarithmic connection ∇_M on M and $DR_{D+E}M$ is a summand of $DR_{D+E}\pi_* O_Y$. Thus (M, ∇) satisfies $E_1(M)$ degeneration since (O_Z, d) satisfier $E_1(O_Z)-$ degeneration. One has $\pi_* O_Y = \bigoplus_{j=0}^{N-1} O(-\{\frac{j}{N}D\})$ and the sheaves $O(-\{\frac{j}{N}D\})$ have logarithmic connections along $D+E$. From E_1 degeneration, the connecting map d_p of long exact sequence is zero, where

$$d_p: H^q(Gr_F^p[p]) \to H^q(Gr_F^{p+1}[p]) = H^q(Gr_F^{p+1}[p+1])$$

is derived from the exact sequence:

$$0 \to Gr_F^{p+1}[p] \to F^p/F^{p+2}[p] \to Gr_F^p[p] \to 0.$$

Just like [EV3, 175], onj has the commutative diagram:

$$\begin{array}{ccc} \boldsymbol{H}^q(DR_{(X-D,E)}O(-\{\frac{j}{N}\}-B)) & \to & H^q(O(-E-\{\frac{j}{N}D\}-B)) \\ \downarrow\beta & & \downarrow \\ \boldsymbol{H}^q(DR_{(X-D,E)}O(-\{\frac{j}{N}D\})) & \xrightarrow{\alpha} & H^q(O(-E-\{\frac{j}{N}D\})). \end{array}$$

From $Gr_F^p H^{p+q}(DR_{(X-D,E)}O(-\{\frac{j}{N}D\})) = H^{p+q}(Gr_F^p DR_{(X-D,E)}O(-\{\frac{j}{N}D\}))$, α is surjective. By the same argument as in [EV3, 167, 171], we conclude that β is isomoprhic, Hence γ is surjective.

Proof of 2). By the same argument as in [EV3, 176], we have the following commutative square:

$$H^q(\Omega_X^p\langle D+E\rangle(-E-\{\tfrac{j}{N}\}))$$

$$\uparrow \qquad\qquad \searrow d_p$$

$$H^q(\Omega_X^p\langle D'+E\rangle(-E-\{\tfrac{j}{N}D\})) \overset{\nabla}{\to} H^q(\Omega_X^{p+1}\langle D+E\rangle(-E-\{\tfrac{j}{N}D\}))$$

$$\text{r} \downarrow \qquad\qquad\qquad\qquad \downarrow \text{residue}$$

$$H^q(\Omega_C^p\langle(D'+E)_C\rangle(-E-\{\tfrac{j}{N}D\})_C)\to H^q(\Omega_C^p\langle(D'+E)_C\rangle(-E-\{\tfrac{j}{N}D\})_C),$$

Here ∇ is a composition with the connecting map and a natural map.
Hence ∇ is zero. Let r be the map induced from the next Kodaira-Spencer sequence:

$$0\to\Omega_X^p\langle D'+C+E\rangle(-E-C)\to\Omega_X{}^p\langle D'+E\rangle(-E) \to \Omega_C^p\langle(D'+E)_C\rangle(-E_C)\to 0.$$

Thus $r=0$ and hence one has the following surjection:

$$\phi_C : H^q(\Omega_X^p\langle D'+C+E\rangle(-E-C-\{\tfrac{j}{N}D\}))\to H^q(\Omega_X^p\langle D'+E\rangle(-E-\{\tfrac{j}{N}D\})).$$

This completes the proof of Theorem 1.

□

Pemark of Theorem 1.

(Esnault-Viehweg's comment)

In the proof of Theorem 1, we have the $E_1(-\{\frac{j}{N}D\}))$-degeneration of the spectral sequence:

$$E_1^{p,q}=H^q(\Omega_X^p\langle D+E\rangle\otimes O(-\{\tfrac{j}{N}D\}))=>\boldsymbol{H}^{p+q}(DR_{D+E}O(-\{\tfrac{j}{N}D\})).$$

By Serre-Verdier duality, we have $E_1(O(\{\frac{j}{N}D\})(-D_{red}-E))$-degeneration of the spectral sequence:

$$E_1^{p,q}=H^q(\Omega_X^p\langle D+E\rangle(-D_{red}-E)\otimes O(\{\tfrac{j}{N}D\}))=>\boldsymbol{H}^{p+q}(DR_{D+E}O(\{\tfrac{j}{N}D\})(-D_{red}-E)),$$

Note that $O(\{\frac{j}{N}D\})(-D_{red})=O(-\{\frac{N-j}{N}D\})$ for $0<j<N$, $\frac{j\nu_i}{N}\in\boldsymbol{Q}\backslash\boldsymbol{Z}$.

Hence we obtain the E_1-degeneration of the spectral sequence:

$$E_1^{p,q}=H^q(\Omega_X^p\langle D+E\rangle(-E)\otimes O(-\{\tfrac{N-j}{N}D\}))=>\boldsymbol{H}^{p+q}(DR_{D+E}O(-\{\tfrac{N+j}{N}D\})(-E))$$

without Deligne-Illusie's ([DI]) nor Arapura's M. H. S. ([A1]).
Thus the maps ${}^t\phi_C : H^q(\Omega_X^p\langle D'+E\rangle(\{\frac{N-j}{N}D\}))\to H^q(\Omega_X^p\langle D'+C+E\rangle(\{\frac{N-j}{N}D\}))$ are injective for all $p, q\geq 0$.

(a) Let V be the local system on $X-(D\cup E)$ defined by $\mathrm{Ker}\nabla_{O(-\{\frac{j}{N}D\})}$.
Letting $j: X-(D\cup E)\subset X-D$ and $\tilde{V}$ a lower or upper canoical extension to X and letting $i: X-D\subset X$ be the canonical immersions, one has

$$H^{p+q}(X-D, E_{X-D}\,;\,\tilde{V})\simeq\boldsymbol{H}^{p+q}(DR_{(X-D,E)}O(-\{\tfrac{j}{N}D\}))\simeq\boldsymbol{H}^{p+q}(\Omega_X^{\cdot}\langle D+E\rangle(-E-\{\tfrac{j}{N}D\}))$$

which are the abutments of the Hoge spectral sequence with "filtration bête" $E_1{}^{p,q}=H^q(\Omega_X^p\langle D+E\rangle(-E-\{\frac{j}{N}D\}))$ for $1\leq j\leq N$.
Moreover, for any divisor B supported in supp $\{\frac{j}{N}D\}$, one has the complexes $DR_{(X-D,E)}O(-\{\frac{j}{N}D\}$

$+B)=\Omega_X^{\cdot}\langle D+E\rangle(-E-\{\frac{j}{N}D\}+B)$, the complexes $DR_{(X-D,E)}O(-\{\frac{j}{N}D\})=\Omega_X^{\cdot}\langle D+E\rangle(-E-\{\frac{j}{N}D\})$ and $\boldsymbol{R}i_*j_!V$ are all quasi-isomorphic. Let $D^b=D_{red}-B_{red}$ and $i^b: X-D\subset X-D^b$ the canonical immersion.

Then $H^k(X-D, E_{X-D}: \tilde{V})\simeq H^k(X-D, j_!V)\simeq \boldsymbol{H}^k(X-D^b, \boldsymbol{R}i_*^bj_!V)\simeq H^k(X-D^b, i_!^bj_!V)$
$\simeq H^k(X-D^b, (E+B)_{(X-D^b)}; \tilde{V})\simeq H^{2n-k}(X-(E\cup B), D^b{}_{X-(E\cup B)}; \tilde{V})\simeq H^{2n-k}(X-E, D^b{}_{X-E}; \tilde{V}^V)$.

Hence $H^{p+q}(X-E, D^b{}_{X-E}; \tilde{V}^V)\simeq \boldsymbol{H}^{p+q}(\Omega_X\langle D+E\rangle(-D^b+\{\frac{j}{N}D\}))$.

(b) Assume that B is effective. Ftom [A1, Lemma2, [D1] and [DI], $\Omega_X^p\langle D^b+E+B\rangle(-E-B)$ is quasi-isomorphic to the complex $\bigoplus_{k\geq 0}(\Omega_{\tilde{B}^k}^p\langle(D^b+E)_{\tilde{B}^k}\rangle(-E_{\tilde{B}^k}))[-k]$, where $\tilde{B}^k$ is the normalization of B^k the union of k-times intersections of distinct components of B. The cohomology $H^{k+q}(\Omega_X^p\langle D^b+E+B\rangle(-E-B-\{\frac{j}{N}D\}))$ is the abutment of the spectral sequence with "filtration bête"

$$E_1{}^{k,q}=H^{k+q}(\tilde{B}^k, \Omega_{\tilde{B}^k}^p\langle(D^b+E)_{\tilde{B}^k}\rangle(-E_{\tilde{B}^k})[-k]).$$

Dually, $H^q(\Omega_X^p\langle D^b+E+B\rangle(-D^b+\{\frac{j}{N}D\}))$ is the abutment of the spectral sequence $E_1{}^{-k,q+k}=H^q(\tilde{B}^k, \Omega_{\tilde{B}^k}^{p-k}\langle(D^b+E)_{\tilde{B}^k}\rangle(-D^b+\{\frac{j}{N}D\})_{\tilde{B}^k})$. Furthermore, $\Omega_X^p\langle D^b+B+E\rangle(-D^b)$ is quasi-isomorphic to $\bigoplus_{k\geq 0}\Omega_{\tilde{B}^k}{}^{p-k}\langle D^b+E\rangle(-D^b)[-k]$.

Corollary 2. (a) *Suppose that some power of L is generated by its global sections. Let E be reduced with only transversal intersections with itself. Let B and B' effective divisors on X such that $B+B'\in|L^k|$ for some $k>0$ and that B and B' intersect transversally with E.*
Then the maps ϕ_B :

$$H^q(L^{-j}(-E-B)) \to H^q(L^{-j}(-E))$$

are surjective for all $q\geq 0$, $j>0$.

(b) *Let D be a semiample $\boldsymbol{Q}$-divisor on X with $\{D\}$ supported in a normal crossing divisor and B an effective divisor such that there exists some effective divisor B' satisfying $B+B'\in|L^k|$* for some *$k>0$ and that B and B' intersect transversally with E. Then*

$$H^q(X, O(K+E+\ulcorner D\urcorner)) \to H^q(X, O(K+E+\ulcorner D\urcorner+B))$$

are *injective for all $q\geq 0$.*

Proof. (a) We may take a blowing up of X in place of X such that $B+B'+E$ has only normal crossings and that E remains reduced and has no common component with $B+B'$. Since L is semiample, we can choose a sufficiently large number λ and a smooth divisor C_λ such thet $L_\lambda=O(C_\lambda)$ and that $C_\lambda+B+B'$ is a divisor with normal crossings only.
From $L^{k+\lambda}=O(B+B'+C_\lambda)$, $L^j(-[\frac{1}{k+\lambda}(B+B'+C_\lambda)])=L^j$ for $\lambda\gg 0$.
Hence Theorem 1 implies that the maps ϕ_B:

$$H^q(L^{-j}(-E-B)) \to H^q(L^{-j}(-E))$$

are surjective.

(b) We state the following Kawamata's Covering lemma.

Lemma [KMM], Th. 1-1-1], *Let X be a non singular variety and D a $\boldsymbol{Q}$-divisor such that the fractional part $\{D\}$ has a support with normal crossings only. Then one can construct a non singular variety and a finite dominant Galois map τ: $X' \to X$ with the Galois group $G = Gal(\mathscr{R}(X')/\mathscr{R}(X))$ such that* (i) $\tau^* D \in \mathrm{Div}(X)$, (ii) $(\tau_* O(\tau^* D))^G = O([D])$, $(\tau^* O(K_{X'} + \tau^* D))^G = O(K_X + \ulcorner D \urcorner)$, *where* $(\ \ )^G$ *means the G-invariant part and G acts naturally on them.*

Applying this lemma and (a), we have injections ${}^t\phi_{\tau^*B}$:

$$H^q(O(K_{X'} + \tau^*(D+E))) \to H^q(O(K_{X'} + \tau^*(D+E+B)).$$

Hence $H^q(O(K_{X'} + \tau^*(D+E))^G) \to H^q(O(K_{X'} + \tau^*(D+E+B)^G)$ are injective. Thus $H^q(O(K_X + \ulcorner D \urcorner + E)) \to H^q(O(K_X + \ulcorner D \urcorner + E + B)$ are injective.

□

Example. The following example of Ramanujam ([R]) is due to T. Fuijta ([F]).

Let M be a one point blow-up $Q_p(\boldsymbol{P}^3)$ of the 3-dimensional projective space $\boldsymbol{P}^3$ and L be a pull-back of a tautological invertible sheaf on $\boldsymbol{P}^3$. Then $H^{1,1}(M, L^{-1}) = H^1(\Omega_M \otimes L^{-1}) = 0$.

Let S be a general member of $|L|$. We have the following exact sequence

$0 \to O_M \otimes L^{-1} \to O_M \to O_S \to 0$,

$0 \to \Omega_M \otimes L^{-1} \to \Omega_M \to \Omega_S \to 0$ and

$0 \to I_S/I_S^2 \to \Omega_{M|S} \to \Omega_S \to 0$.

Thus, using $I_S/I_S^2 = O_S(1)$, $S \simeq \boldsymbol{P}^2$,

we have the long exact sequence:

$0 \to H^0(\Omega_{M|S}) \to H^0(\Omega_S) \to H^1(O_S(1)) \to H^1(\Omega_{M|S}) \to H^2(O_S(1))$ Here $H^1(O_S(1)) = H^2(O_S(1)) = 0$, $H^1(\Omega_S) = \boldsymbol{C}$. Hence $H^1(\Omega_{M|S}) = \boldsymbol{C}$.

Using $H^0(\Omega_{M|S}) = H^0(\Omega_S) = 0$ and the following long exact sequence $\cdots \to H^0(\Omega_{M|S}) \to H^1(\Omega_M \otimes L^{-1}) \to H^1(\Omega_M) \to H^1(\Omega_{M|S}) \to \cdots$, we obtain $H^1(\Omega_M \otimes L^{-1}) = \boldsymbol{C}$. since $H^1(\Omega_M) = \boldsymbol{C}^2$, $H^1(\Omega_{M \equiv S}) = \boldsymbol{C}$.

Let μ be the birational map from $Q_p(\boldsymbol{P}^3)$ onto $\boldsymbol{P}^3$ and $E = Q_p(p)$. N. Nakayama poits out that $R^2 \mu_* O(K_{Q_p(P_3)} + E)$ is not torsion free. In fact, from $0 \to O(K_Q) \to O(K_Q + E) \to O_E(K_E) \to 0$, one gets $0 = R^2 \mu_* O(K_Q) \to R^2 \mu_* O(K_Q + E) \to R^2 \mu_* O_E(K_E) \to 0$.

Definition. Let X be a locally noetherian scheme, F a coherent O_X-Module. F is said to be without associated embedded prime component if Ass(F) consists of maximal elements only. F is said to be irredvndant if Ass(F) consists of a single element.

Definition. Let $f: X' \to X$ be a morphism of ringed spaces. Let $\mathscr{S}_f$ be a sheaf of germs of regular sections of θ_X such that its images are also regular by the homomorphism $\Gamma(U, O_X) \to \Gamma(f^{-1}U, O_{X'})$ for any open set of X. We denote $\theta_X[\mathscr{S}_f^{-1}]$ by $\mathscr{M}_f$. We say that the pull-back of a divisor D on X exists if there exist $s_D \in \Gamma(X, \mathscr{M}_f(O_X(D)))$ and $s_{-D} \in \Gamma(X, \mathscr{M}_f(O_X(-D)))$. We denote by f^*D the couple $(f^*D, s_D \circ f)$ and f^*D is called the pull-back of D.

A divisor D is called f-effective if there exists $s_D \in \Gamma(X, \mathscr{S}_f(O_X(D)))$ and $D = \mathrm{div}\ s_D$.

Theorem 3. Logarithmic Kollár-type vanishing ([Kol], [EVII])

Let X be a compact complex manifold bimeromorphically dominated by a Kähler manifold, L an invertible sheaf and E an effective reduced divisor suppoted in a normal crossing divisor on X. Let $f: X \to S$ be a morphism into a projective $\boldsymbol{C}$-scheme. Let D be an effective divisor on X supported in a normal crossing divisor. Suppose that E has only transversal intersections with the pull-back of *any divisor os S and suppese that $D+E$ is supported in a normal crossing divisor and that E and D have no common component.*

Let K be a nef big divisor on S with respect to f, i. e. the sum of K and any ample divisor is always ample and for any f-effective divisor B there exists an f-effective divisor B' such that $O(B+B')=K^{\lambda}$ for some $\lambda>0$ and that f^*K exists.

Assume that $(L^N(-D))^b=f^*K$ for some $N>1$, $b>0$. Given any f-effective divisor B on S, *the maps*

$$\phi_{f^*B}: H^q(L^{-j}([\frac{j}{N}D]-E)(-f^*B))\to H^q(L^{-j}([\frac{j}{N}D]-E))$$

are surjective for all $q\geq 0$, $1\leq j$.

Proof. Let A be an ample divisor on S. There exist an f-effective divisor F and $c>0$ such that $O(cK)=O(A+F)$. Since K is nef big, $tK+A$ is ample for any $t>0$. From $(L^N(-D))^b=f^*K$, $(L^N(-D))^{bt}(f^*A)=O(f^*(tK+A))$ is semiample for all $t>0$. Using $O(A)=O(cK-F)$, we have $(L^N(-D))^{bt}(f^*(cU-F))=(L^N(-D))^{b(t+c)}(-f^*F)=f^*(O(tK+A))$. For each $t>0$, there exists a number $k>0$ such that $O(k(tK+A))$ is very ample.

There exists an f-effective divisor B' such that $O(B+B')=O(c'K)$ for some $c'>0$.

We may replace X by a blowing up of X such that $f^*(kF+B+B')$ is a divisor with normal crossings only.

We take a non singular divisor C_k from $|f^*(Y(k(tK+A)))|$ such that C_k has transversal intersection only with $D+f^*(B+B'+kF)$.

We have

$L^{Nb(k(t+c)+c')}=O(b(k(t+c)+c')D+f^*(B+B'+kF)+C_k)$. Hence,

$$L^j(-[\frac{j}{Nb(k(t+c)+c')}((b(k(t+c)+c')D+f^*(B+B'+kF)+C_k)]=L^j(-[\frac{j}{N}-D])$$

for large $t>0$.

Note that $f^*B\subset\text{supp}\ \{\frac{j}{Nb(k(t+c+c'))}(b(k(t+c)+c')D+f^*(B+B'+kF)+C)\}$.

Hence we have the following commutative triangle:

$$\begin{array}{ccc} H^q(L^{-j}([\frac{j}{N}D]-E-f^*(B+B'))) & \to & H^q(L^{-j}([\frac{j}{N}D]-E) \\ \searrow & & \nearrow \\ & H^q(L^{-j}([\frac{j}{N}D]-E-f^*B)). & \end{array}$$

Thus we complete the proof from Theorem 1.

□

Remark 3. 1. *In the proof of Theorem 3, we just assume that $f^*(B+B'+kF)$ intersect transversally with E such that $O(cK)=O(A+F)$, $O(c'K)=O(B+B')$.*

Remark 3. 2. *In Theorem 3, if we take $D=0$, $L=f^*K$, then the maps ${}^t\phi_{f^*B}: H^q(L^j(K_X+E))\to H^q(L^j(K_X+E+f^*B))$ are injective for all $q\geq 0$, $j>1$.*

Corollary 4. *Assume the same hypothesis as in Theorem 3.*

(a) *Logarithmic Kollar vanishing theorem*

(i) $H^i(R^qf_*O(K_X+E+L^j(-[\frac{j}{N}D]))=0$ *for all* $q\geq 0$, $i, j\geq 1$.

(ii) $H^q(O(K_X+E+L^j(-[\frac{j}{N}D])))=H^0(R^qf_*O(K_X+E+L^j(-[\frac{j}{N}D])))$ *for all* $q\geq 0, j>0$.

(iii) *Let $g: S\to T$ be a generically finite morphism of projective schemes. Then* $R^qg_*R^pf_*O(K_X$

$+E+L^j(-[\frac{j}{N}D])=0$ *for all* $q>0,\ p\geq 0,\ j\geq 0.$

(b) $R^qf_*O(K_X+E+L^j(-[\frac{j}{N}D]))$ *are irredundant for* $j\geq 1.$

Proof. Since S is noetherian, Ass $(R^qf_*O(K_X+E)\otimes L^j(-[\frac{j}{N}D]))$ consists of a finite number of elements. One take a very ample divisos A such that the multiplications by $\psi\in\Gamma(\mathscr{S}_fO(A))$ where ψ_x does not belong to j_x for any associated point $x\in$ Ass $R^qf_*O(K_X+E)\otimes L^j(-[\frac{j}{N}D])$

$$R^qf_*O(K_X+E+L^j(-[\frac{j}{N}D]))\rightarrow R^qf_*O(K_X+E+L_j(-[\frac{j}{N}D])(A)$$

are injective and that

$$H^i(R^qf_*O(K_X+E+L^j(-[\frac{j}{N}D])(A))=0 \text{ for all } i>0.$$

We may assume, moreover, that $V=f^{-1}(A)$ is non singular.

Hence from the exact sequence $0\rightarrow O_X(-V)\rightarrow O_X\rightarrow O_V\rightarrow 0$, and the induction hypothesis

$$H^i(R^qf_*O_V(K_V+E+L^j(-[\frac{j}{N}D]))_V)=0 \quad \text{for all } i>0,$$

we have

$$H^{i+1}(R^qf_*O(K_X+E+L^i(-[\frac{j}{N}D])))=H^i(R^qf_*O_V(K_V+E+L^j(-[\frac{j}{N}D]))_V)=0 \text{ for all } i>0.$$

If f is totally ranified, we replace S by the Stein-factorization. In order to prove the vanishing of $H^1(R^qf_*O(K_X+E+L^j(-[\frac{j}{N}D])))$ we use the following Leray spectral sequence

$$E_2^{i,q}=H^i(R^qf_*O(K_X+E+L^j(-[\frac{j}{N}D])))=\rangle E^{i+q}=H^q(O(K_X+E+L^j(-[\frac{j}{N}D]))).$$

Let $r=Min\{q|E_2^{1,q}\neq 0)$, *Then* $E_2^{1,r}=E_\infty^{1,r}\subset E^{r+1}$, since $E_2^{1,q}=0$ for $i\geq 2.$

Thus we have the following commutative square:

$$\begin{array}{ccc} E_1^{2,r}=H^1(R^rf_*O(K_X+E+L^j(-[\frac{j}{N}D])))\subset & E^{1+r}=H^{1+r}(O(K_X+E+L^j(-[\frac{j}{N}D]))) \\ \downarrow & \downarrow \\ E_2^{1,r}=H^1(R^rf_*O(K_X+E+L^j(-[\frac{j}{N}D](A))))\subset & E^{1+r}=H^{1+r}(O(K_X+E+L^j(-[\frac{j}{N}D])(f^*A))) \end{array}$$

From Serre vanishing and injectivity of the left vertical arrow, $H^1(R^rf_*O(K_X+E+L^j(-[\frac{j}{N}D])))$ $=0$, which contradicts the minimality.

(b) Let x be a point of Ass $(R^qf_*O(K_X+E+L^j(-[\frac{j}{N}D])))$ such that there exists an f-effective divisor B satisfying that $\psi_x\in j_x$ for $\psi\in\Gamma(\mathscr{S}_f(B))$. Let A be an ample invertible sheaf on S such that $R^qf_*O(K_X+E+L^j(-[\frac{j}{N}D]))\otimes A$ is generated by its global sections.

Applying Theorom 3 to $\hat{L}=L\otimes f^*A$ and (a) of Corollary 4, we have the commutative square:

$$\begin{array}{ccc} H^0(R^qf_*O(K_X+E+\hat{L}^j(-[\frac{j}{N}D]))=H^q(O(K_X+E+\hat{L}^j(-[\frac{j}{N}D]))) \\ \downarrow \qquad\qquad \downarrow \\ H^0(R^qf_*O(K_X+E+\hat{L}^j(-[\frac{j}{N}D])(f^*B)))=H^0(R^qf_*O(K_X+E+\hat{L}^j(-[\frac{j}{N}D])(f^*B))). \end{array}$$

Thus the injectivity of the vertical arrows of the above diagram contradicts the existence of such an associated point x.

Thus Ass $(R^q f_* O(K_X+E+L^j(-[\frac{j}{N}D])))$ consists of a maximal element x and the closure of x in just the image $f(X)$.

We complete the proof. □

Remark 4. 1. In (a) of Corollary 4, it suffices to assume that there exists an f-effective divisor F such that $f^*(F)$ intersects transversally with E and $A+F=cK$ for some $c>0$ and a general ample divisor A passing through $f(X)$.

Remark 4. 2. In (b) of Corollary 4, any associated point of $R^q f_* L^j(-[\frac{j}{N}D]+K_X+E)$ will be found at the point x such that there exist f-effective divisors B and B' such that $\psi_x \in j_x$ for $\psi \in \Gamma(\mathscr{S}_f O(B))$ and that $f^*(B+B')$ has tangential contacts with E.

We explain the definition and basics about the generalized normal crossing variety developed by Kawamata [Ka2] as it is.

Definion. A reduced equi-dimensional projective scheme X is called a generalized normal crossing variety if the complete local rings of $O_{X,x}$ for the points x of X are isomorphic to

$k(x)[[x_{01}, \cdots, x_{0r}]]\otimes \bigotimes_{1\leq i\leq t} k(x)[[x_{i1}, \cdots, x_{ir_i}]]/(x_{i1}, \cdots, x_{ir_i})$, where the integest t and r_1 depend on x and $k(x)=O_x/j_x$.

If $t\leq 1$ for all x, then X is called a normal crossing variety, A generalized normal crossing variety X is locally complete intersection and hence has an invertible dualizing sheaf ω_X. We denote by K_X div s where $s \in \Gamma(X, \mathscr{M}(\omega_X)^*)$. Let X_0 be the normalization of X and let $X.$ be a simplicial scheme given by

$$\Delta_n \to X_n = X_0 \times_X X_0 \cdots \times_X X_0 \quad (n+1 \text{ factors})$$

The projection $\varepsilon.: X. \to X$ gives a hypercovering [D3]. Note that X_n are non singular. The union of B_n on the X_n of the images of lower dimensional irreducible components of the X_{n}, for $n'>n$ form divisors with only normal crossings on the X_n. A divisor D on X is called permissible if it induces divisors D_n on the X_n when pulled back by $\varepsilon_n: X_n \to X$, It is equivalent to saying that the support of D does not contain any stratum of X locally. We denote by $\mathrm{Div}_0 X$ the group of permissible divisors. A generalized normal crossing divistor D on X is a permissible divisor such that the unions $D_n \cup B_n$ are reduced divisors with normal crossings only on the X_n. This condition holds if and only if the local equation of D at $x \in X$ is given by $x_{01} \cdots x_{0s_0}=0$ for some $s_0 \leq r_0$. It is important for D to become also a generalized normal crossing variety. Let D be an element of $\mathrm{Div}_0(X)\otimes \boldsymbol{Q}$ whose support is a generalized normal crossing divisor. Then we can define a permissible devisor $\ulcorner D \urcorner$ by the system of divisors $\ulcorner D_n \urcorner$ on the X_n. A birational morphisr $\mu: Y \to X$ of generalized normal crossing varieties is called permissible with respect to D if (a) C induces non singular subschemes C_n on the X_n, (b) an arbitrary irreducible component of the C_n does not coincide with a whole irreducible component of the X_n and if (c) the centers C_n on the X_n are permissible with respect to the $D_n \cup B_n$ in the sense of Hironaka [H].

A permissible blowing up gives a permissible birational morphism of generalized normal crossing varieties and the union of the inverse images of C and the strict transform of D is again a generalized normal crossing divisor. A permissible birational morphism can be dominated by a succession of permissible blowing ups [H]. Let $\varepsilon._* O_X$ be the cochain complex whose coboundaries are given by alternating sums of face operators. Then $\varepsilon.$ induces a quasiisomorphism $O_X \simeq \varepsilon._* O_X.$ (Ka2] , 575). Let D be an effective permissible divisor on X and D_n the pull-backs of D on the X_n. Then we have a quasi-isomorphism $O_D \simeq \varepsilon._* O_D.$ ([Ka2], 576).

We shall prove the following

Theorem 5. *Let X be a generalized normal crossing variety and $D \in \mathrm{Div}_0(X) \otimes \boldsymbol{Q}$, $E \in \mathrm{Div}_0(X)$. Assume that D is semi-ample and $\{D\}$ is supported in a generalized normal crossing divisor. Let E be an effective reduced divisor on X supported in a generalized normal crossing divisor. Let B an effective divisor $\in \mathrm{Div}_0(X)$ such that there exists an effective divisor $B' \in \mathrm{Div}_0(X)$ such that $O(B+B') = L^\lambda$ for some $\lambda > 0$. Suppose that $B+B'$ intersects transversally with E. Then*

(a) *the maps*

$$\phi_B : H^q(O(K_X+E+\ulcorner D \urcorner)) \to H^q(O(K_X+E+B+\ulcorner D \urcorner))$$

are injective atm all $q \geq 0$.

(b) *Let S be a projcetiue scheme given by the image of the morphism associated with the surjection $O_X \otimes \Gamma(O_X(nD)) \to O_X(nD)$ for some $n > 0$.*

Let $f : X \to S$ denote the merphism above.

$H^r(R^q f_* O(K_X+E+\ulcorner D \urcorner)) = 0$ for $r > 0$, $q \geq 0$.

Hence $H^q(O(K_X+E+\ulcorner D \urcorner) = H^0(R^q f_*(O_X(K_X+E+\ulcorner D \urcorner))$.

(c) *$R^q f_*(O(K_X+E+\ulcorner D \urcorner)$ is without associated embedded prime component if any effective divisor $H \in |kD|$ for some $k > 0$ intersects tranrversally with E and if describing $E = \sum E_i$ we have no component E_j such that $f(E_j) \subsetneqq f(E_i)$ for some i.*

Proof. From preliminaries explained above, we have the following commutative diagram:

$$\begin{array}{ccc} E_1^{-p,q+p} = H^{q+p}(X_p, O_{X_p}(K_{X_p}+E_p+\ulcorner D_p \urcorner) & \Rightarrow & H^q(X, O(K_X+E+\ulcorner D \urcorner) \\ \uparrow \text{zero} & & \uparrow \text{zero} \\ {}'E^{-p,q-1+p} = H^{q-1+p}(B_p, O_{B_p}(K_{B_p}+E_p+\ulcorner D_p \urcorner) & \Rightarrow & H^{q-1}(B, O(K_B+E+\ulcorner D_B \urcorner)) \end{array}$$

Since the connecting map ${}'E^{-p,q-1+p} \to E^{-p,q+p}$ for all p, q are zero, the connecting map of the abutment is also zero. □

We shall show Kodaira-Nakano-Norimatsu type vanishing.

Theorem 6. *Let X be a projective complex manifold and L an invertible sheaf of X. Let E, D be effective divisors on X with $E+D$ supported in normal crossing divisors with no common component. Suppose that $L^N(-D)$ is an ample invertible sheaf of X for some $N > 1$. Then $H^q(\Omega_X^p\langle D+E\rangle(-D_{red}) \otimes L^j(-[\frac{j}{N}D])) = 0$ for $p+q > \dim X$, $j \geq 1$.*

Proof. We proceed by induction with respect to dimension X.

The first step is obvious. There exists a number k such that $(L^N(-D))^k$ is very ample. We choose a smooth divistr $C \in |(L^N(-D))^k|$ which intersects transversally with $D+E$. Note that $L^{kN} = O(kD+C)$ and $(kD+C)_{red} - C = D_{red}$. From Theorem 1. the maps

$${}^t\phi_C : H^q(\Omega_X^p\langle D+E\rangle(-D_{red}) \otimes L^j(-[\frac{j}{N}D])) \to H^q(\Omega_X^p\langle D+E+C\rangle(-D_{red}) \otimes L^j(-[\frac{j}{N}D]))$$

are injective for all p, q. By indution hypothesis,

$$H^{q-1}(\Omega_C^p\langle(D+E)_C\rangle(-D_{red})_C\otimes O(C|_C)\otimes L_C^j(-[\frac{j}{N}D_C]))=0 \text{ for } p+q-1>\dim C.$$

Thus, from a Kodaira-Spencer sequence

$$0\to\Omega_X^p\langle D+E+C\rangle(-D_{red}-C)\to\Omega_X^p\langle D+E\rangle(-D_{red})\to\Omega_C^p\langle(D+E)_C\rangle(-D_{red})_C\to 0$$

the maps ϕ_C :

$$H^q(\Omega_X^p\langle D+E+C\rangle(-D_{red})\otimes L^j(-[\frac{j}{N}D]))\to H^q(\Omega_X^p\langle D+E\rangle(C-D_{red})\otimes L^j(-[\frac{j}{N}D]))$$

are injective ay induction hypothesis for $p+q>$din X.

We can choose an ample divisor C such that the latter cohomology groups vanish by Serre's Theorem. Therefore we complete the proof.

□

Theorem 7. *Let X be a compact complex manifold bimeromorphically dominated by a Kähler manifold, L an invertible sheaf on X and D, E effective divisors on X. Suppose that some power of $L^N(-D)$ for $N>1$ is generated by its global sections. Let $f:X\to S$ be a dominant morphism associated with a surjection $O_X\otimes\Gamma((L^N(-D))^n)\to(L^N(-D))^n$ for some $n>0$.*

(a) *Let B be an effective divisor on S such that $O(f^*B)=(L^N(-D))^b$ for some number b. Suppose that f^*B is irreducible and smooth and that $D+E+f^*B$ is supported in a normal crossing divisor with no common component one another.*

Then the maps ϕ_{f^*B} :

$$H^q(\Omega_X^p\langle D+E\rangle(-D_{red})\otimes L^j(-[\frac{j}{N}D]))\to H^q(\Omega_X^p\langle D+E+f^*B\rangle(-D_{red})\otimes L^j(-[\frac{j}{N}D]))$$

are all injective for p, $q\geq 0$, $j>0$.

(a) *Assume tiat the generic point of a general ample divisor A* in *S is not an associated point of*

$$R^qf_*\Omega_X^p\langle D+E+f^{-1}(A)\rangle(-D_{red})\otimes L^j(-[\frac{j}{N}D]).$$

Then

(i) $H^q(\Omega_X^p\langle D+E\rangle(-D_{red})\otimes L^j(-[\frac{j}{N}D]))\simeq H^0(R^qf_*\Omega^p\langle D+E\rangle(-D_{red})\otimes L^j(-[\frac{j}{N}D]))$ *for* $j\geq 1$, p, $q\geq 0$.

(ii) $H^i(R^qf_*\Omega_X^p\langle D+E\rangle(-D_{red})\otimes L^i(-[\frac{j}{N}D]))=0$ *for* $i>0$, $j\geq 1$, p, $q\geq 0$,

Proof. (a) Since $(L^N(-D))^b=O(f^*B)$, it follows that

$$L^j(-[\frac{j}{Nb}(bD+f^*B)]=L^j(-[\frac{j}{N}D]).$$

Noting that $(bD+f^*B)_{red}=D+f^*B$ and that f^*B is irreducible and smooth, from Theorem 1, we obtain the injectivity of the maps

$$\phi_{f^*B}:\ H^q(\Omega_X^p\langle D+E\rangle(-D_{red})\otimes L^j(-[\frac{j}{N}D]))\to H^q(\Omega_X^p\langle D+E+f^*B\rangle(-D_{red})\otimes L^j(-[\frac{j}{N}D]))$$

for p, $q\geq 0$.

Proof of (b). Since S is noetherian, Ass $(R^qf_*\Omega_X^p\langle D+E\rangle(-D_{red})\otimes L^j(-[\frac{j}{N}D]))$ consists of a finite number of elements. One can take a very ample divisor A such that (i) the multiplications by $\psi\in\Gamma(\mathscr{S}_fO(A))$ where ψ_X does not belong to j_X for any associated point $x\in$ Ass $R^qf_*\Omega_X^p\langle D+E\rangle(-D_{red})\otimes L^j(-[\frac{j}{N}D])$:

$$R^qf_*\Omega_X^p\langle D+E\rangle(-D_{red})\otimes L^j(-[\frac{j}{N}D])\overset{\psi}{\to}R^qf_*\Omega_X^p\langle D+E\rangle(-D_{red})\otimes L^j(-[\frac{j}{N}D])(A)$$

are injective and

(iii) $H^i(R^qf_*\Omega_X^p\langle D+E\rangle(-D_{red})\otimes(-[\frac{j}{N}D])(A))=0$ for all $i>0$.

We may assume, mteover, that $V=f^{-1}(A)$ is non singular.

We have the following commutative triangle:

$$\begin{array}{ccc}
0 & & R^qf_*\Omega_V^p\langle(D+E)_V\rangle(-D_{red}+V)_V\otimes L_V{}^{(j)}\\
\downarrow & & \uparrow\\
R^qf_*\Omega^p\langle D+E\rangle(-D_{red})\otimes L^{(j)} & \to & R^qf_*\Omega^p\langle D+E\rangle(-D_{red}+V)\otimes L^{(j)}\\
\downarrow & & \\
R^qf_*\Omega^p\langle D+E+V\rangle(-D_{red})\otimes L^{(j)}\ \nearrow & & \\
\downarrow & & \\
R^qf_*\Omega_V^{p-1}\langle(D+E)_V\rangle(-D_{red})_V\otimes L_V{}^{(j)} & &
\end{array}$$

Here we denote $L^j(-[\frac{j}{N}D])$ by $L^{(j)}$ as in [EV3].

By the choice of A, the horizontal arrow is injective. Hence from the commutativity of the above triangle the slant arrow is injective. Hence from the commutativity of the above triangle the slant arrow in also injective. Hence applying H^i to the diagram above, one has

$$\begin{array}{ccc}
0 & & H^iR^qF_*\Omega_V^p\langle(D+E)_V\rangle(-D_{red}+V)_V\otimes L_V{}^{(j)}\\
\downarrow & & \uparrow\\
H^iR^qF_*\Omega^p\langle D+E\rangle(-D_{red})\otimes L^{(j)} & \to & H^iR^qf_*\Omega^p\langle D+E\rangle(-D_{red}+V)\otimes L^{(j)}\\
\downarrow & & \\
H^iR^qf_*\Omega^p\langle D+E+V\rangle(-D_{red})\otimes L^{(j)}\ \nearrow & & \\
\downarrow & & \\
H^iR^qf_*\Omega_V{}^{p-1}\langle(D+E)_V\rangle(-D_{red})_V\otimes L_V{}^{(j)} & &
\end{array}$$

The left vertical sequence in exact. From induction hypothesis,

$$H^iR^qf_*\Omega_V{}^{p-1}\langle D+E)_V\rangle(-D_{red})_V\otimes L_V{}^{(j)}=0 \text{ for } i>0.$$

Hence $H^iR^qf_*\Omega_X^p\langle D+E\rangle(-D_{red})\otimes L^{(j)}\simeq H^iR^qf_*\Omega_X^p\langle D+E+V\rangle(-D_{red})\otimes L^{(j)}$ for $i>2$.

By hypothesis, the slant upwards sequence is exact. Hence from induction assumption $H^i(R^qf_*\Omega_V{}^p\langle(D+E)_V\rangle(-D_{red}+V)_V\otimes L_V{}^{(j)}=0$,
$H^i(R^qf_*\Omega_X{}^p\langle D+E+V\rangle(-D_{red})\otimes L^{(j)})\simeq H^i(R^qf_*\Omega_X{}^p\langle D+E\rangle(-D_{red}+V)\otimes L^{(j)})=0$.
In order to prove the vanishing of $H^1(R^qf_*\Omega_X\langle D+E\rangle(-D_{red})\otimes L^j(-[\frac{j}{N}D]))$, we use the following Leray spectral sequence

$$E_2{}^{i,q}=H^i(R^qf_*\Omega_X{}^p\langle D+E\rangle(-D_{red})\otimes L^j(-[\frac{j}{N}D]))$$
$$=>E^{i+q}=H^q(\Omega_X\langle D+E\rangle(-D_{red})\otimes L^j(-[\frac{j}{N}D])).$$

Let $r=\min\{q|E_2{}^{1,q}\neq 0\}$. Then $E_2{}^{1,r}=E_\infty{}^{1,r}\subset E^{r+1}$, since $E_2{}^{i,q}=0$ fet $i>1$.

Thus we have the following commutative square:

$$E_2^{1,r}=H^1R^qf_*\Omega_X\langle D+E\rangle(-D^r)\otimes L^j(-[\frac{j}{N}D])\subset E^{1+r}=H^{1+r}(\Omega_X\langle D+E\rangle(-D^r)\otimes L^j(-[\frac{j}{N}D]))$$
$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$
$$H^1(R^rf_*\Omega_X{}^p\langle D+E\rangle(-D^r)\otimes L^j(-[\frac{j}{N}D])(A)))\subset H^{1+r}(\Omega_X{}^p\langle D+E\rangle(-D^r)\otimes L^j(-[\frac{j}{N}D])|f^*A)).$$

From Serre vanishing, $H^1(R^rf_*\Omega_X\langle D+E\rangle(-D^r)\otimes L^j(-[\frac{j}{N}D]))=0$, which contradicts the minimality of r.

We complete the proof.

Corollary 8.

(a) *Let* $r=$ *the maximal dimension of* f, *i. e.,* $Max\{\dim f^{-1}(s)|s\in S\}$, *for* $p+q>\dim X+r$. *One has*

$$H^q(\Omega_X^p\langle D+E\rangle(-D_{red})\otimes L^j(-[\frac{j}{N}D]))=0.$$

(b) *Logarithmic Arapura vanishing*

Suppose moreover that $D_{red}=f^{-1}(Z)$ *for some closed set* Z *in* S *and that the restriction* f *to* $X-D$ *is equidimensional. Then for* $p+q>2\cdot\dim X-\dim S$, *one has*

$$H^q(\Omega_X^p\langle D+E\rangle(-D_{red})\otimes L^j(-[\frac{j}{N}D]))=0.$$

(c) $R^qf_*\Omega_X{}^p\langle D+E\rangle(-D_{sed})\otimes L^j(-[\frac{j}{N}D])=0$ *for* $p+q>\dim X+r$.

moreover, under the assumption of (b)

$$R^qf_*\Omega_X{}^p\langle D+E\rangle(-D_{red})\otimes L^j(-[\frac{j}{N}D])=0$$

for $p+q>2\cdot\dim X-\dim S$.

Proof. It suffices to prove (b)

Let B be an ample divisor satisfying the condition of Theorem 7.

Hence the map $\phi_f{}^*{}_B$ is injective. On the other hand,

$$H^{p+q}(\Omega_X\langle D+E+f^*B\rangle(-D_{red}\rangle\otimes L^j(-[\frac{j}{N}D]))\simeq H^{p+q}(X-f^*B, j_!(\Omega_X\langle E\rangle\otimes L^j(-[\frac{j}{N}D])))$$

is the abutment of

$$E_2{}^{q,p}\simeq H^q(\text{affine}, j_!R^pf|_{X-D*}(\Omega_X\langle E\rangle\otimes L^j(-[\frac{j}{N}D])))\simeq H^q(\text{affine}, J_!R^pf|_{X-D*}(V))=0$$

for $q>\dim S$ or $p>2\cdot(\dim X-\dim S)$,
i. e., $p+q>2\cdot\dim X-\dim x$.

Thus we complete the proof.

□

The above Themrem implies a partial Bogomolov-Sommese vanishing.

Theorem 9. Bogomolov-Sommese vanithing ([B], [BS], [BV3])

Let X *be a compact complex manifold bimeromorphically dominated by a Kähler manifold,* L *an invertible sheaf and* E *an effective divisor supported in a normal crossing divisor. Then if* $\kappa(L)>p$,

$$H^0(\Omega_X^p\langle E\rangle\otimes L^{-j})=0 \text{ for } j>0.$$

Proof. The statement may be proved after replacing by a dominant manifold. From $\kappa(L)>p$, there exists a number N and effective divisors H, F such that $L^N=O(H+F)$ and that H is base free with $\kappa(O(H))=\kappa(L)$. We may assume that $H+F$ is a normal crossing divisor.

Taking Kawamta covering $\tau: Y\to X$ as to $\frac{1}{N}(H+F)$, $\tau^*\frac{1}{N}(H+F)=\tau^*L$. Let $L'=O(\tau^*\frac{1}{N}H)$. Assume that the statement is right for L'. Then $0=H^0(\Omega_Y^p\otimes L'^{-j})\supset H^0(\Omega_X^p\otimes L^{-j})$. It suffices to show the statement for a semiample L. Thus for some $N>1$, $L^N=O(C)$ for a smooth divisor C with transversal intersections only with E. From Theorem 1, the map $\phi_C: H^0(\Omega_X{}^p\langle C+E\rangle(-C)\otimes L^{-j})\to H^0(\Omega_X{}^p\langle E\rangle\otimes L^{-j})$ is surjective. On the other hand, the exact sequence

$$0\to H^0(\Omega_X{}^p\langle E\rangle\otimes L^{-N-j})\to H^0(\Omega_X^p\langle E+C\rangle\otimes L^{-N-j})\to H^0(\Omega_C{}^{p-1}\langle E_C\rangle\otimes L_C{}^{-N-j})$$

gives the isomorphism of the first two terms from induction assumption that $H^0(\Omega_C{}^{p-1}\langle E_C\rangle\otimes L^{-N-j})=0$ by $\kappa(L_C)>p-1$.

If necessary, replacing ay a larger N, we obtain the vanishing $H^0(\Omega_X^p\langle E\rangle\otimes L^{-N-j})=0$ from numerical criterion on the associated projective bundle $\boldsymbol{P}(\Omega_X^p\langle E\rangle\otimes L^{-N-j})$.

Thus we complete the proof.

□

Theorem 10. *Let X be a compact complex manifold bimeromorphdcally dominated by a Kähler manifold and L an invertible sheaf on X.*

Let $f: X\to S$ be a morphism into a projective $\boldsymbol{C}$-scheme.

Let D, E be effective divisors on X with $D+E$ supported in normal crossing divisors without common componts.

Assume the spectral sequnce $E_1{}^{k,q}=H^q(\tilde{B}^k, \Omega_{\tilde{B}^k}^p\langle(D'+E)_{\tilde{B}^k}\rangle(-E)_{\tilde{B}^k}\otimes L_{\tilde{B}^k}{}^{(j)})$

$$E^{k+q}=H^{k+q}(\Omega_X^p\langle D'+E+B\rangle(-E-B)\otimes L^{(j)}) \text{ fot } D'=D_{red}-B,$$

any reduced divisor $B\in$ supp $\{\frac{j}{N}D\}$ degenerates at $E_2{}^{k,q}$.

Suppose that E has only transversal intersections with the pull-back of any divisor on S. Let K be a nef big divisor on S with respect to f (see Theorem 2) *such that for an ample divisor A, $O(cK)=O(A+F)$ for some f-effective divisor F and $c>0$ and that the sum of K and arbitrary ample divisor is also ample. Assume that $(L^N(-D))^b=f^*K$ for some $N>1$, $b>0$. Given any f-effective divisor B on such that $f^*(B)$ is smooth and that there exists an f-effective divisor B' such that $O(B+B')=O(c'K)$ for some $c'>0$ and that $f^*(B+B'+F)$ is supported in a normal crossing divisor, then the maps*

$$\phi_{f^*B}: H^q(\Omega_X^p\langle D+E\rangle(-D_{red})\otimes L^j(-[\tfrac{j}{N}D]))\to H^q(\Omega_X^p\langle D+E+C\rangle(-D_{red})\otimes(-[\tfrac{j}{N}D]))$$

are all injective for $p, q\geq 0$.

Proof. Let A be an ample divisor on S. There exist an f-effective divisor F and $c>0$ such that $O(cK)=O(A+F)$. Since K is nef big, $tK+A$ is ample for any $t>0$. From $(L^M(-D))^b=f^*K$, $(L^N(-D))^{bt}(f^*A)=O(f^*(tK+A))$ is semiample for all $t>0$. Using $O(A)=O(cK-F)$, we have $(L^N(-D))^{bt}(f^*(cK-F))=(L^N(-D))^{b(t+c)}(-f^*F)=f^*(O(tK+A))$. For each $t>0$, there exists a number $k>0$ such that $O(k(tK+A))$ is very ample. There exists an f-effective divisor B' such that $O(B+B')=O(c'K)$ for some $c'>0$.

From assumption, $f^*(F+B+B')$ is a divisor with normal crossings only. We take a smooth divisor C_k from $|f^*(O(k(tK+A)))|$ such that C_k intersects transversally with $D+E+f^*(B+B'+kF)$. $f^*(B+B'+kF)+C_k$ being described by Δ_k, $D+\Delta_k$ is a divisor with normal crossings only. We have $L^{Nb(k(t+c)+c')}=O(b(k(t+c)+c')D+f^*(B+B'+kF)+C_k)$.

Hence,

$$L^j(\frac{j}{Nb(k(t+c)-c')}((b(k(t+c)+c')D+f^*(B+B'+kF)+C_k)=L^j(-[\frac{j}{N}D])$$

for large $t>0$.

Note that $f^*B\subset\operatorname{supp}\{\frac{j}{Nb(k(t+c)+c')}(b(k(t+c)+c')D+f^*(B+B'+kF)+C_k)\}$.

Hence we have the following commutative triangle:

$$\begin{array}{ccc} H^q(\Omega_X^p\langle D+E\rangle(-D_{red})\otimes L^j(-[\frac{j}{N}D])) & \to & H^q(\Omega_X^p\langle D+E+f^*(B+B')\rangle(-D_{red})\otimes L^j(-[\frac{j}{N}D])) \\ \searrow & & \nearrow \\ & H^q(\Omega_X^p\langle D+E+f^*B\rangle(-D_{red})\otimes L^j(-[\frac{j}{N}D])). & \end{array}$$

Thus we complete the proof.

□

Lemma 11.

Let X be a compact complex bimeromorphically dominated by a Kähler manifold, L an invertible sheaf on X and D effective divisors w!ith $D+E$ supported in a normal crossing divisor on X and without any common component. Assume that E is reduced. Let $E=E^1+E^2$ be a decomposition without common component. Let $L^N=O(D)$ for some $N>1$. Let $1\leq j\leq N-1$.

(i) *Let B be an effective reduced divisor supported in* supp D *and $i^b: X-B\subset X$ a canonical immersion. For the sake of simplicity we denote i^b by i here. Let $i^m{}_*O_{X-B}$ denote the sections meromorphic along B ([D5]). Let $\sum_{1\leq i\leq\nu} B_i$ be the irreducible decomposition of B. Let $\beta\in\boldsymbol{Z}^\nu$. Let ${}^\beta B$ denote $\sum_{1\leq i\leq\nu}\beta_iB_i$. We have a decomposition $B=B^1+B^2$ without common component.*

Assume that B is supported in $\{\frac{j}{N}D\}$ for $1\leq j\leq N-1$ when B^1 does not appear. We may assume that just B^1 is supported in $\{\frac{j}{N}D\}$ when B^1 in explicitly described.

The complexes $i_!i^*\Omega_X^\cdot\langle D'+E\rangle(-E^1-\{\frac{j}{N}D\})$, $Ri_*i^*\Omega_X^\cdot\langle D'+E\rangle(-E^1-\{\frac{j}{N}D\})$ and $\Omega_X\langle D'+B+E\rangle(-E^1+{}^\beta B-\{\frac{j}{N}D\})$ are quasi-isomorphisms for all $\beta\in\boldsymbol{Z}^\nu$.

By Verdier duality, the complexes $Ri_*i\Omega_X^\cdot\langle D'+E\rangle(-E^2-D'+\{\frac{j}{N}D\})$,

$i_!i^*\Omega_X\langle D'+E\rangle(-E^2-D'+\{\frac{j}{N}D\})$ and $\Omega_X^\cdot\langle D'+B+E\rangle(D'-E^2+{}^\beta B+\{\frac{j}{N}D\})$

are quasi-isomorphic for all $\beta\in\boldsymbol{Z}^\nu$.

(ii) *From* (i), *we have the following isomorpisms:*

$$\boldsymbol{H}^{p+q}(X-B,\Omega_X^\cdot\langle D'+E\rangle(-E^1-\{\frac{j}{N}D\}))\simeq\boldsymbol{H}^{p+q}(X,\Omega_X^\cdot\langle D+E\rangle(-E^1-{}^\beta B-\{\frac{j}{N}D\})$$

$$\simeq\boldsymbol{H}_{X-B}^{p+q}(X-B,\Omega_X^\cdot\langle D'+E\rangle(-E^1-\{\frac{j}{N}D\}))$$

$$\simeq \boldsymbol{H}^{p+q}(X,\ i_*^m i^* \Omega_X^{\cdot}\langle D'+E\rangle(-E^1-{}^{\beta}B-\{\tfrac{j}{N}D\}))\ \textit{and}$$

$$\boldsymbol{H}^{p+q}(X-B,\ \Omega_{X-B}^{\cdot}\langle D'+E\rangle(-E^2-D'+\{\tfrac{j}{N}D\}))$$

$$\simeq \boldsymbol{H}^{p+q}(X,\ \Omega_X^{\cdot}\langle D+E\rangle(-E^2-D'+{}^{\beta}B+\{\tfrac{j}{N}D\}))$$

$$\simeq \boldsymbol{H}_{X-B}^{p+q}(X,\ \Omega_{X-B}^{\cdot}\langle D'+E\rangle(-E^2-D'+\{\tfrac{j}{N}D\}))$$

$$\simeq \boldsymbol{H}^{p+q}(X,\ i_*^m i^* \Omega_X\langle D'+E\rangle(-E^2-D'+\{\tfrac{j}{N}D\}))$$

for all $\beta \in \boldsymbol{Z}^{\nu}$.

These cohomologies have Mixed Hodge structures.

(iii) *The spectral sequence with "filtration bête", i. e., Hodge spectral sequences*:

$$E_1^{p,q}=H^{p+q}(X,\ \Omega^p\langle D+E\rangle(-E^1-B-\{\tfrac{j}{N}D\})[-p])\Longrightarrow$$

$$E^{p,q}=\boldsymbol{H}^{p+q}(X,\ \Omega_X^{\cdot}\langle D+E\rangle(-E^1-{}^{\beta}B-\{\tfrac{j}{N}D\}))$$

$$E_1^{p,q}=H^{p+q}(X,\ \Omega_X^p\langle D+E\rangle(-E^2-D'+\{\tfrac{j}{N}D\})\Longrightarrow$$

$$E^{p,q}=\boldsymbol{H}^{p+q}(X,\ \Omega_X^{\cdot}\langle D+E\rangle(-E^2-D'+{}^{\beta}B-\{\tfrac{j}{N}D\}))\ \textit{for}\ \beta\in\boldsymbol{Z}^{\nu},\ 1\leq j<N,$$

degenerate at $E_1^{\cdot,\cdot}$. *There exist the spectral sequences*

$$E_1^{p,q}=H^{p+q}(X-B,\ \Omega^p\langle D'+E\rangle(-E^1-\{\tfrac{j}{N}D\})[-p])\Longrightarrow$$

$$E^{p,q}=\boldsymbol{H}^{p+q}(X-B,\ \Omega^{\cdot}\langle D'+E\rangle(-E^1-\{\tfrac{j}{N}D\}))\ \textit{and}$$

$$E_1^{p,q}=H_{X-B}^{p+q}(\Omega_{X-B}^p\langle D'+E\rangle(-E^1-\{\tfrac{j}{N}D\}))\Longrightarrow$$

$$E^{p,q}=\boldsymbol{H}_{X-B}{}^{p+q}(X,\ \Omega_{X-B}^{\cdot}\langle D'+E\rangle(-E^1-\{\tfrac{j}{N}D\}))$$

(iii)′ *Let P be the pole order filtration of*

$i_*^m i^* \Omega_X^{\cdot}\langle D+E\rangle(-E^1-B-\{\tfrac{j}{N}D\})$

i. e., $P^t(\Omega^{\cdot}\langle D+E\rangle(-E^1-B-\{\tfrac{j}{N}D\}))$

$$=\bigoplus_p P^{t-p}(i_*^m O)\otimes\Omega^p\langle D'+E\rangle(-E^1-B-\{\tfrac{j}{N}D\})[-p].$$

Then from [D1].

$$H^q(X,\ \Omega^p\langle D+E\rangle(-E^1-B-\{\tfrac{j}{N}D\}))=H^q(X,\ Gr_P^p\Omega^{\cdot}\langle D'+E\rangle(-E^1-B-\{\tfrac{j}{N}D\})).$$

(iv) *The spectral sequences*:

$$E_1^{k,q}=H^q((\tilde{B}^1)^k,\ \Omega^p{}_{(\tilde{B}1)^k}\langle D'+E\rangle(-E^1-B^2-\{\tfrac{j}{N}D\}))$$

$$\Longrightarrow E^{k+q}=H^{k+q}(X,\ \Omega_X^p\langle D+E\rangle(-E^1-B-\{\tfrac{j}{N}D\}))$$

and

$$E_1^{-k,q+k}=H^q((\tilde{B}^1)^k,\ \Omega^p_{(\tilde{B}^1)^k}\langle D'+B^2+E\rangle(-D'+\{\frac{j}{N}D\}))\Longrightarrow$$

$$E^q=H^q(X,\ \Omega^p_X\langle D+E\rangle(-D'+\{\frac{j}{N}D\}))$$

djgenerate at $E_2^{\cdot,\cdot}$.

(v) $H^q(X,\ \Omega^p_X\langle D+E\rangle(-E^1-B-\{\frac{j}{N}D\})\to H^q(X,\ \Omega^p_X\langle D'+B^2+E\rangle(-E^1-B^2-\{\frac{j}{N}D\})$

are all surjective for $p,\ q\geq 0$.

$H^q(X,\ \Omega^p_X\langle D+B^2+E\rangle(-E^2-D'+\{\frac{j}{N}D\}))\to H^q(X,\ \Omega^p_X\langle D+E\rangle(-E^2-D'+\{\frac{j}{N}D\}))$

are all injective for $p,\ q\geq 0$.

(vi) *Let* $f: X\to S$ *be a morphism into a projective* $\boldsymbol{C}$*-scheme.*

$Rf_*\Omega^{\cdot}_X\langle D+E\rangle(-E^1-{}^{\beta}B-\{\frac{j}{N}D\})\simeq R(f\circ i)_*\Omega^{\cdot}_{X-B}\langle D'+E\rangle(-E^1-\{\frac{j}{N}D\})$

$\simeq Rf_*i_!\Omega^{\cdot}_{X-B}\langle D'+E\rangle(-E^1-\{\frac{j}{N}D\})$

(vii)

$\mathscr{E}_1^{p,q}=R^qf_*\Omega^p_X\langle D+E\rangle(-D'-E^2+\{\frac{j}{N}D\})$

${}'\mathscr{E}_1^{p,q}=R^q(f\circ i)_*\Omega^p_X\langle D+E\rangle(-D'-E^2+\{\frac{j}{N}D\})$

${}''\mathscr{E}_1^{p,q}=R^qf_*i_!\Omega^p_{X-B}\langle D+E\rangle(-D'-E^2+\{\frac{j}{N}D\})$

$\Longrightarrow \boldsymbol{R}^{p+q}f_*\Omega^{\cdot}_X\langle D+E\rangle(-D'-E^2+{}^{\beta}B+\{\frac{j}{N}D\})$

degenerate at $\mathscr{E}_1^{\cdot,\cdot}$.

except ${}'\mathscr{E}_1^{\cdot,\cdot}$, ${}''\mathscr{E}_1^{\cdot,\cdot}$ *for* $\beta\in\boldsymbol{Z}^{\nu}$.

(viii) *The spectral sequences*:

$\mathscr{E}_1^{-k,q+k}=R^qf_*\Omega^{p-k}_{(\tilde{B}^1)^k}\langle D'+B^2+E\rangle(-D'-E^2+\{\frac{j}{N}D\})\Longrightarrow$

$\mathscr{E}^q=R^qf_*\Omega^p_X\langle D+E\rangle(-D'-E^2+\{\frac{j}{N}D\}))$

degenerate at $\mathscr{E}_2^{\cdot,\cdot}$.

(ix)

$R^qf_*\Omega_X^p\langle D'+B^2+E\rangle(-D'+\{\frac{j}{N}D\})\to R^qf_*\Omega^p_X\langle D+E\rangle(-D'+\{\frac{j}{N}D\})$

are all injective for $p,\ q\geq 0$.

Proof. (i)$\sim$(iii)$'$ except (iii).

We denote the canonical immersions by

Fig.
$$\begin{array}{ccc} X-B & \underset{i}{\subset} & X \\ \cup k' & & \cup\ k \\ X-(B+D') & \underset{i'}{\subset} & U''=X-D' \\ \cup j' & & \cup\ j \\ U=X-(B+D'+E) & \underset{i''}{\subset} & U'=X-(D'+E)\ . \end{array}$$

We refer to the following ([EV3], lemma (1.6)),

Lemma. *Let V be a local system on U such that the mondromies γ_j of V around B_j for $j=1, \cdots, \nu$ do not have* 1 *as eigenvalues. Then $i_! V$ and $Ri_* V$ are quasi-isomorphic. Furthermore $R^q i_* V=0$ for $q>0$.*

Proof. ([EV3]) It suffices to show this locally. We may assume that $W=\prod_{j=1}^{n}\Delta_j$ and $U\cap W=\prod_{j=1}^{r}\Delta_j\times\prod_{j=r+1}^{n}\Delta_j^*$, where the Δ_j are small disks and Δ_j^* are the punctured disks. The monodromies γ_j around $W\cap B$ commute. There exists a local subsystem V' of V stable by the γ_j such that the cokernel is also a local system. Hence we are reduced to a local system V of rank one. We may write $V=\bigotimes_{j=r+1}^{n} pr_j^* V_j$, where $pr_j: U\cap W\to\Delta_j^*$ is the j-th projection and V_j the local system on Δ_j^* associated to the representation γ_j on a vecter space L of rank one. By Künneth formula, $H^k(\Delta_j^*, V_j)=0$ for $k=0, k=1$.

Replacing Δ_j^* by $S^1\sim\partial\Delta_j^*$ and parametrizing S^1 by $e^{2\pi it}$ for $t\in\boldsymbol{R}$, we have a covering $\{U_1, U_2\}$ of S^1, where $U_1=\{e^{2\pi it}|t\in(0,1)\}$ and $U_2=\{e^{2\pi is}|s\in(\frac{-1}{2},\frac{1}{2})\}$. Then U_1, U_2 and the two connected componenst W^+, W^- are simply connected. The coordinate change from U_1 to U_2 is

$$W^+\cup W^-\to W^+\cup W^-$$

$$t\to s=\begin{cases} t & \text{if } t\in W^+ \\ t-1 & \text{if } t\in W^-\end{cases}$$

The Čech cohomology with values in V_j is computed by the cohomology of the complex

$$0\to L_{U_1}\times L_{U_2}\xrightarrow{d} L_{W^+}\times L_{W^-}\to 0$$

$$(l_1, l_2)\to(l_1-l_2, l_1-\gamma_j l_2)=(l_1, l_2)\begin{Vmatrix} 1 & 1 \\ -1 & -\gamma_j\end{Vmatrix}$$

When $\gamma_j\neq 1$, d is an isomorphism.

Q. D. D.

From this lemma ([EV3], [D1]) and with $\{\frac{j}{N}D\}$ not having one as eigenvalue the complexes $i_!''\Omega_U^{\cdot}(-\{\frac{j}{N}D\})$, $i_*''\Omega_U^{\cdot}(-\{\frac{j}{N}\})$, $i_*''\Omega_U^{\cdot}(-\{\frac{j}{N}D\})$, $i_*''^m\Omega_U^{\cdot}(-\{\frac{j}{N}D\})$, $\Omega_{U'}^{\cdot}\langle B_{U'}\rangle(-\{\frac{j}{N}D\})$, $\Omega_{U'}^{\cdot}\langle B_{U'}\rangle(-^{\beta}B-\{\frac{j}{N}D\})$ for any $\beta\ni\boldsymbol{Z}^{\nu}$ are quasi-isomorphic.

From the commutativity of the diagram above, say,

$$i_!i^*\Omega_X^{\cdot}\langle D'+E\rangle(-E-\{\tfrac{j}{N}\})=i_!\Omega_{X-B}^{\cdot}\langle D'+E\rangle(-E-\{\tfrac{j}{N}D\})$$

$$=i_!k'_*\Omega_{X-D}^{\cdot}\langle E\rangle(-E-\{\tfrac{j}{N}D\})=i_!k'_*j_!'\Omega_U^{\cdot}(-\{\tfrac{j}{N}D\})=k_*j_!i_*''\Omega_U^{\cdot}(-\{\tfrac{j}{N}D\})$$

$$=k_*j_!\Omega_{U'}^{\cdot}\langle B_{U'}\rangle(-^{\beta}B-\{\tfrac{j}{N}D\})_{U''}=k_*\Omega_{U''}^{\cdot}\langle(E+B)_{U''}\rangle(-^{\beta}B-\{\tfrac{j}{N}D\})_{U''}$$

$$=\Omega_X^{\cdot}\langle D'+B+E\rangle(-^{\beta}B-\{\tfrac{j}{N}D\})$$

in the derived category. Taking Verdier dual, we have

$$\boldsymbol{D}(i_!i^*\Omega_X^{\cdot}\langle D'+E\rangle(-E-\{\tfrac{j}{N}D\}))=i_*i^*\Omega_X^{\cdot}\langle D'+E\rangle(-D'+\{\tfrac{j}{N}D\}).$$

From [D1, prop. 3.13 p. 80], a morphism of filtered complexes

$$(\Omega_X^{\cdot}\langle D+E\rangle(-E^1-^{\beta}B-\{\tfrac{j}{N}D\}), F)\cong(i_*{}^m i\Omega_X^{\cdot}\langle D'+E\rangle(-E^1-^{\beta}B-\{\tfrac{j}{N}D\}), P)$$

is filtered quasi-isomorphism. Hence

$$E_1^{p,q}=H^q(\Omega_X^p\langle D+E\rangle(-E^1-{}^{\beta}B-\{\tfrac{j}{N}D\}))\simeq H^{p+q}(Gr_P{}^p i_*{}^m i^*\Omega_X^{\cdot}\langle D'+E\rangle(-E^1-{}^{\beta}B-\{\tfrac{j}{N}D\})).$$

(iii), (iv)

From Deligne([D2], 22), recall the notation

$$E_1^{p,n-p-q}:=H^n(Gr_F^p Gr_W^q K)\simeq H^n(Gr_W^q Gr_F^p K).$$

Fig.

$$\begin{array}{ccccc} & & {}_WE_1^{q,d}=H^n Gr_W^q K & & \\ & \nearrow & & \searrow & \\ E_1^{q,q,n-p-q}=H^n Gr_{pF} Gr_W^q K & & & & H^n K \\ & \searrow & & \nearrow & \\ & & {}_FE_1^{p,n}=H^n Gr_F^p K & & \end{array}$$

Let $W(B^1)$ be an increasing weight filtration with respect to a divisor B^1, i. s.

$$W(B)_n(\Omega^p\langle D+E\rangle(-E^2-D'+\{\tfrac{j}{N}D\}))$$
$$=\Omega^n\langle D+E\rangle(-E^2-D'+\{\tfrac{j}{N}D\})\wedge\Omega^{p-n}\langle D'+B^2+E\rangle(-E^2-D'+\{\tfrac{j}{N}D\}).$$

We denote by ε^n a local system of rank 1 which indicates the orientation of $(B^1)^n$([D2]).

Let $i_n:(\tilde{B}^1)^n\to X$ be the canonical maps. Taking Poincaré residue, we have

$$Gr_n^{W(B^1)}\Omega_X^{\cdot}\langle D+E\rangle(-E^2-D'+\{\tfrac{j}{N}D\})\simeq i_{n*}\Omega_{(\tilde{B}1)^n}\langle D'+E\rangle(\varepsilon^n-E^2-D'+\{\tfrac{j}{N}D\})[-n].$$

We have the spectral sequence

$${}_WE_1^{+n,k+n}=\boldsymbol{H}^k(X,\ Gr_n^{W(B^1)}\Omega_X^{\cdot}\langle D+E\rangle(-E^2-D'+\{\tfrac{j}{N}D\}))$$
$$\simeq\boldsymbol{H}^{k-n}((\tilde{B})^n,\ \Omega^{\cdot}_{(\tilde{B}1)^n}\langle D'+B^2+E\rangle(\varepsilon^n)(-E^2-D'+\{\tfrac{j}{N}D\}))$$
$$\Longrightarrow E^k=\boldsymbol{H}^k(X,\ \Omega_X^{\cdot}\langle D+E\rangle(-E^2-D'+\{\tfrac{j}{N}D\})).$$

Then the differential maps

$d_r:{}_WE_r^{p,q}\to{}_WE_r^{p+r,q-r+1}$ are strictly compatible with the induced filtration F_r for $r\geq 0$([D2], (3.2.6)). In fact, The spectral sequence

$$E_1^{p,n,k-p+n}=H^k(Gr_F^p Gr_n^{W(B^1)}\Omega_X^{\cdot}\langle D+E\rangle(-E^2-B-D'+\{\tfrac{j}{N}D\})[-p])$$
$$\simeq H^{k-p-n}(\Omega^p{}_{(\tilde{B}1)^n}\langle D'+B^2+E\rangle(\varepsilon^n-E^2-B^2-D'+\{\tfrac{j}{N}D\}))$$
$$\Longrightarrow{}_WE_1^{-n,k+n}=\boldsymbol{H}^{k-n}(\Omega_{(\tilde{B}1)^n}\langle D'+B^2+E\rangle(\varepsilon^n-E^2-B^2-D'+\{\tfrac{j}{N}D\}))$$
$$\simeq H^{k-n}((\tilde{B}^1)^n-(D+E-B^1)_{(\tilde{B}1)^n},\ (E^2+B^2+D')_{(\tilde{B}1)^n};\ V^{\vee}{}_{(\tilde{B}1)^n}(\varepsilon^n))$$

fixing n, degenerates at E_1 by induction hypothesis on the number of components of B and the same argument as in Theorem 1 when $B=\phi$. Furthermore the filtration of the abutment above is $p+q+n$ opposed with its complex conjugate. From[D2, prop. 132], E_1-degeneration is equivalent to the strict compatibility with the filtration.

Fer $r\geq 2$, by induction hypothesis and([D2], 1.3.16」, we may assume that an induced filtration F_r in independent of orders of sub-quotients on ${}_WE_s^{\cdot,\cdot}$ for $s\leq r+1$ and that ${}_WE_r^{\cdot,\cdot}={}_WE_2^{\cdot,\cdot}$.

The induced filtration F_r is q-opposed to the complex conjugate $\bar{F}_r$.

Thus

$$d({}_wE_r^{p,q})=d_r(\sum_{a+b=q}F_r^a({}_wE_r^{p,q})\cap\bar{F}_r^b({}_wE_r^{p,q}))\subset\sum_{a+b=q}F^a({}_wE_r^{p+r,q-r+1})\cap\bar{F}^b({}_wE_r^{p+r,q-r+1})=0.$$

Hence ${}_wE_2^{p,q}$ degenerates.

We have the required spectral sequence

$$\begin{aligned}E_1^{p,-n,q+n}&\simeq H^q((Gr_n^{W(B^1)}\Omega_X^p\langle D+E\rangle(-E^2-D'+\{\tfrac{j}{N}D\}))\\&\simeq H^q(\Omega_{(\tilde{B}1)^n}^{p-n}\langle D'+B^2+E\rangle(\varepsilon^n)(-E^2-D'+\{\tfrac{j}{N}D\}))\\&\Longrightarrow {}_FE^{p,q}=H^q(\Omega_X^p\langle D+E\rangle(-E^2-D'+\{\tfrac{j}{N}D\})).\end{aligned}$$

Fig.
$$\begin{array}{ccccc} & & H^{p+q}Gr_{\bar{W}}^{-n}K & & \\ & \nearrow & & \searrow & \\ H^{p+q}Gr_F^pGr_{\bar{W}}^{-n}K & & & & H^{p+q}K \\ & \searrow & & \nearrow & \\ & & H^{p+q}Gr_F^pK & & \end{array}$$

The same argument of Corollary(3.2.23) in [D2] gives the degenenration of $E_2^{p,-n,q+n}$ fixing p. In fact,

$$\begin{aligned}\textstyle\sum_k \dim E^k&=\textstyle\sum_k \dim \boldsymbol{H}^k(X,\Omega_X^{\cdot}\langle D+E\rangle(-E^2-D'+\{\tfrac{j}{N}D\}))\\&=\textstyle\sum_{k,n}\dim {}_wE_2^{-n,k+n}=\sum_{k,n,p}\dim E_2^{p,-n,k+n-p}.\end{aligned}$$

On the other hand,

$$\textstyle\sum_k \dim E^k\leq\sum_{k,p}\dim {}_FE^{p,k-p}\leq\sum_{k,p,n}\dim E_2^{p,-n,k+n-p}.$$

Hence two spectral sequences

$$\begin{aligned}E_1^{p,-n,k-p+n}&\simeq H^{k-p}(Gr_n^{W(B^1)}\Omega_X^p\langle D+E\rangle(-E^2-D'+\{\tfrac{j}{N}D\}))\\&\simeq H^{k-p}(\Omega_{(\tilde{B}^1)^n}^{p-n}\langle D'+B^2+E\rangle(\varepsilon^n)(-E^2-D'+\{\tfrac{j}{N}D\}))\\&\Longrightarrow H^{k-p}(\Omega_X^p\langle D+E\rangle(-E^2-D'+\{\tfrac{j}{N}D\}))={}_FE_1^{p,k-p}\text{ fixing }p,\end{aligned}$$

$$E_1^{p,k-p}\Longrightarrow E^k=\boldsymbol{H}^k(\mathrm{X},\Omega_X^{\cdot}\langle D+E\rangle(-E^2-D'+\{\tfrac{j}{N}D\}))$$

degenerate at $E_2^{p,-n,k-p+n}$ and ${}_FE_1^{p,k-p}$, respectively.

Another degeneration in obtained by Verdier duality. Thus we get (iv).

(v)

$$\begin{aligned}E_2^{0,q}&=H(E_1^{-1,q}\to E_1^{0,q}\to E_1^{0,q+1})\\&=H(0\to H^q(\Omega_X^p\langle D'+B^2\rangle(-E^1-B^2-\{\tfrac{j}{N}D\}))\\&\qquad\to H^q(\Omega_{(\tilde{B}1)^1}^p\langle D'+B^2+E\rangle(-E^1-B^2-\{\tfrac{j}{N}D\}))\\&=\ker(H^q(\Omega_X^p\langle D'+B^2+E\rangle(-E^1-B^2-\{\tfrac{j}{N}D\}))\\&\qquad\to H^q(\Omega_{(\tilde{B}1)^1}^p\langle D'+B^2+E\rangle(-E^1-B^2-\{\tfrac{j}{N}D\}))\\&=H^q(\Omega_X^p\langle D'+B^2+E\rangle(-E^1-B^2-\{\tfrac{j}{N}D\})))\end{aligned}$$

The last equality is derived by the same argument of the remark in ([EV3, p177). Indeed,

$$\mathrm{res}_{(\tilde{B}1)^1}\circ\nabla O(-\{\frac{j}{N}D\}) \to \Omega_X{}^1\langle D+E\rangle(-\{\frac{j}{N}D\}) \to O_{(\tilde{B}1)^1}(D'+B^2+E-\{\frac{j}{N}D\})$$

is surjective. And we have E_1-degeneration of $\boldsymbol{H}^{p+q}(X, \Omega^{\cdot}\langle D+E\rangle(-E^1-B^2-\{\frac{j}{N}D)))$.

Since $E^{k,q}=0$ for $k<0$, we have the surjections
$E^q \to E_\infty{}^{0,q}=E_2{}^{0,q}$, i. e.,

$$H^q(\Omega_X{}^p\langle D'+B^1+B^2+E\rangle(-B^1-B^2-E^1-\{\frac{j}{N}D\})) \to H^q(\Omega_X^p\langle D'+B^2+E\rangle(-E^1-B^2-\{\frac{j}{N}D\})).$$

Dually,

$$H^q(\Omega_X{}^p\langle D'+B^2+E\rangle(-E^2-D'+\{\frac{j}{N}D\})) \to H^q(\Omega_X{}^p\langle D+E\rangle(-E^2-D'+\{\frac{j}{N}D\}))$$

are injective for all p, q.

(vi)～(ix)

From (i), (vi) in a direct consequence. As for (vii)～(ix), we take a sufficiently ample invertible sheaf A on S.

Let $\hat{L}=L\otimes f^*A$.

$$\bigoplus_{p+q=n} H^q(\Omega_X{}^p\langle D+E\rangle(-E^2-D'+\{\frac{j}{N}D\})\otimes f^*A)\simeq \boldsymbol{H}^n(\Omega_X^{\cdot}\langle D+E\rangle(-E^2-D'+\{\frac{j}{N}D\})\otimes f^*A)$$

$$\bigoplus_{p+q=n} H^0(R^qf_*\Omega_X{}^p\langle D+E\rangle(-E^2-D'+(\frac{j}{N}D\})\otimes A)$$

$$\simeq H^0(\boldsymbol{R}^nf_*\Omega_X^{\cdot}\langle D+E\rangle(-E^2-D'+\{\frac{j}{N}D\})\otimes A)$$

Thus we get (vii).

Similarly,

$$E_2{}^{-k,q+k}=H(E_1{}^{-k-1,q-1+k+1} \to E_1{}^{-k,q+k} \to E_1{}^{-k+1,q+1+k-1})$$

$$H(H^0(R^{q-1}f_*\Omega^{p-k-1}_{(\tilde{B}^1)^{k+1}}\langle D'+B^2+E\rangle(-E^2-D'+\{\frac{j}{N}D\})\otimes A)) \to$$

$$H^0(R^qf_*\Omega^{p-k}_{(\tilde{B}^1)^{k-1}}\langle D'+B^2+E\rangle(-E^2-D'+\{\frac{j}{N}D\})\otimes A))$$

$$\to H^0(R^{q+1}f_*\Omega^{q-k+1}_{(\tilde{B}^1)^{k-1}}\langle D'+B^2+E\rangle(-E^2-D'+\{\frac{j}{N}D\})\otimes A))$$

$$=H(H^0(\mathscr{E}_1{}^{-k-1,k+q}\otimes A) \to H^0(\mathscr{E}_1{}^{-k,q+k}\otimes A) \to H^0(\mathscr{E}_1{}^{-k+1,q+k}\otimes A)$$

$$=H^0(\mathscr{E}_2{}^{-k,q+k}\otimes A)=H^0(\mathscr{E}_\infty{}^{-k,q+k}\otimes A).$$

Thus we get (viii) and (ix).

□

Theorem 12.

Let X be a compact complex manifold bimeromorphcally dominated by a Kähler manifold and S a projective complex variety. Let $f: X\to S$ be a proper surjective morphism from X onto S. Let L be an invertible sheaf on X and D and E effective divisors on X. Assume that D and E are supported in a normal crossing divisor and have no common components. Furthermore, let $L^N=O(D)$ for $N>1$ and let E be a reduced divisor. Let M be an ample invertible sheaf on S such that M is generated by its global sections, Then $R^qf_*\omega_X(E+\{\frac{j}{N}D\})\otimes M^k$ for $1\leq j<N$, dim *$S<k$, $0\leq q$ are generically generated by its global sections.*

Proof. We shall prove in the same way as in [EV2], [Ko1], [V2].

Let A be a general member of the linear system $|M|$ such that f^*A is a non singular divisor on X, which we denote by V. We have the next exact sequence:

$$0\to R^qf_*\omega_X(E+\{\frac{j}{N}D\})\otimes M^k(-A)\to R^qf_*\omega_X(E+\{\frac{j}{N}D\})\otimes M^k\to R^qf_*\omega_V(E+\{\frac{j}{N}D\})\otimes M_A{}^{k-1}\to 0$$

Here the injection is given by multiplying the defining section of A. Note that $R^qf_*\omega_V(E+\{\frac{j}{N}D\})\otimes M_A{}^k=R^qf_*\omega_X(E+\{\frac{j}{N}D\})\otimes M^k|_A$. From Theorem 6,

$H^1(S, R^qf_*\omega_X(E+\{\frac{j}{N}D\})\otimes M^{k-1})=0$. Hence

$$H^0(S, R^qf_*\omega_X(E+\{\frac{j}{N}D\})\otimes M^k)\to H^0(A, R^qf_*\omega_V(E+\{\frac{j}{N}D\})\otimes M_A{}^{k-1})$$

are surjective. Thus by induction argument, $R^qf_*\omega_X(E+\{\frac{j}{N}D\})\otimes M^k$ are generically generated by its global sections. □

Corollary 13. *In addition we assume that X and S are projective manifolds.*

(i) $(R^qf_*\omega_{X/S}(E+\{\frac{j}{N}D\}))^{\otimes s}\otimes M^k$ *are generically generated by its global sections for* $1\leq j<N$, $\dim S<k$, $q=0$, $s>0$.

(ii) $(f_*(\omega_{X/S}(E))^l)^{\otimes s}\otimes M^k$ *are generically generated by its global sections for* $l\geq 1$, $s>0$, $k>\dim S$.

(iii) $f_*(\omega_{X/S}(E))^l$ *are weakly 1-positve for* $l>0$.

Proof.

Replacing S by some open dense set S° of S with codim $(S-S^\circ, S)\geq 2$ and f by the restriction $f|_{f^{-1}(S^\circ)}: X\cap f^{-1}(S^\circ)\to S^\circ$

we may assume that $f: X\to S$ is flat.

By X^s we denote $X\times_S\cdots\times_S X$ and by $\mu: X^{(s)}\to X^s$ a desingularization.

Then from [V], $R^qf^{(s)}\omega X^{(s)}{}_{/S}(\mu^*\pi_S^s(E+\{\frac{j}{N}D\}))\to(R^qf_*\omega_{X/S}(E+\{\frac{j}{N}D\}))^{\otimes s}$ are generically isomorphic.

Thus $R^qf_*\omega_{X/S}(E+\{\frac{j}{N}D\})\otimes M^k$ are generically generated by its global sections.

(iii) We may assume that $f: X\to S$ is already replaced by $f^\circ: f^{-1}(S^\circ)\to S^\circ$ for an open subset of S with codim $(S-S^\circ, S)\geq 2$ such that $f_*\omega_{X/S}(E)^l|_{S^\circ}$ is locally free and that f° is projective and flat.

Replacing X by a blowing-up of X, we have

$$Im(f^*f_*(\omega_{X/S}(E))^l\to\omega_{X/S}{}^l)=\omega_{X/S}{}^l(-F)$$

for some effective divisor F supported in a normal crossing divisor.

Put $L=\omega_{X/S}(E)^{l-1}\otimes f^*M^k$. Then $L^l(-(l-1)F)=(\omega_{X/S}(E)^l(-F))^{l-1}\otimes f^*M^{kl}$.

From [M], $f_*\omega_{X/S}(E)^l$ are weakly 1-positive; hence $(f_*\omega_{X/S}(E)^l)^{\alpha\beta}\otimes M^{k\beta}$ are generically generated by its global sections for any $\beta\geq\beta_0$. Take $N=\beta l$ for a fixed β.

Thus $f_*(L(-[\frac{l-1}{l}F])\otimes\omega_{X/S}(E))=f_*(\omega_{X/S}(E)^l(-[\frac{l-1}{l}F]))\otimes M^k=f_*\omega_{X/S}(E)^l\otimes_{O_S}M^k$.

Hence the natural injection $f_*(L^{(1)}\otimes\omega_{X/S}(E))\to f_*(L(-[\frac{l-1}{l}F])\otimes\omega_{X/S})$ is generically isomorphic. From (ii) the first term is generically generated by its global sections. Thus we ger (iii).

□

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