# An odd pair of axioms bring semigroups fairly near to the group

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It is well-known that a semigroup is a group if it has a right-identity and right-inverses. A question was raised by H. Suzuki as to whether it is the case if a semigroup has a *right*-identity and *left*-inverses. The present note shows that although it is impossible to exclude oddities, these oddities can be expelled into a plain direct-product factor.

## 1. Introduction

It is well-known that a semigroup $S$ is a group if it satisfies the two additional axioms, namely the existence of a right-identity $e$ and that of right-inverses w. r. t. $e$. In this paper we postulate the existence of a *right*-identity $e$ and that of *left*-inverses w. r. t. *each* right-identity $e$, and claim that under these axioms emerge semigroups which are just the direct product of the semigroup with the "*right-forgetting*" binary operation and a group.

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## 2. The definition and result

For a semigroup $S$, we put $E:=\{e\in S;\ ^{\forall}x\in S,\ xe=x\}$, and call $S$ a *right-left semigroup* (abbrev. *RL-semigroup*) if it satisfies the following two conditions:

(1) $E\neq\phi$;

(2) $^{\forall}x\in S,\ ^{\forall}e\in E,\ \exists y\in S;\ yx=e.$

Examples:

1) A group $G$ is an RL-semigroup.

2) Any set $E\ (\neq\phi)$ equipped with the operation

$$xy=x\ (^{\forall}x,\ y\in E)$$

is an RL-semigroup.

3) If $S$ and $T$ are RL-semigroups, so is the direct product semigroup $S\times T$.

Our goal is to show that by these examples are exhausted all RL-semigroups, namely,

Theorem. *If S is an RL-semigroup, then it is isomorphic to the direct product of a semigroup E with the operation given in the example 2) and a group G.*

## 3. Proof

First we make a list of a few properties of RL-semigroups which are deduced straightforwardly from the axioms (1) and (2).

(3) $^{\forall}x\in S,\ \exists y\in S;\ yx\in E$

(4) if $xy\in E$ then $yx\in E$

(5) $^{\forall}x\in S,\ \exists y\in S;\ xy\in E$

(6) if $ax=bx$, then $a=b$

(7) if $e,\ f\in E$, then $ef\in E$

Property (3) follows from (2). To prove (4), let $z$ be an element such that $zx\in E$. Then for any $u\in S$, we have

$$u(yx)=u(zx)(yx)=(uz)(xy)x$$

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$= (uz)x = u(zx) = u.$

Property (5) is obvious from (3) and (4), and (6) is straight from (5). (7) follows from the definition of $E$. Note that thus $E$ is a subsemigroup of $S$.

Definition. A relation $\sim$ on $S$ is defined by that $x \sim y$ iff $x = ey$ for some $e \in E$.

We claim:

(8) $x \sim x$

(9) if $x \sim y$, then $y \sim x$

(10) if $x \sim y$, $y \sim z$, then $x \sim z$

Property (8) follows from (3) and (4); (10) from (7). To prove (9), we let $x = ey$ $(e \in E)$, and let $z$ be an element such that $yz \in E$. Then,

$$xz = (ey)x = e(yz) = e \in E$$

so that $zx \in E$ by (4), and

$$y = y(zx) = (yz)x,$$

which shows that $y \sim x$. Thus $\sim$ is an equivalence relation, and we put $G = S/\sim := \{[x];\ x \in S\}$, the set of equivalence classes $[x] := \{y \in S;\ y \sim x\}$.

The following three properties are trivial by definition:

(11) if $e$, $f \in E$, then $e \sim f$

(12) more generally, if $e$, $f \in E$ and $x \in S$, then $ex \sim fx$

(13) if $x \sim u$ and $y \sim v$, then $xy \sim uv$

Definition. We define an operation on the set $G$ by putting

$$[x][y] := [xy],$$

for all $[x], [y] \in G$. That this operation is well-defined is guaranteed by the property (13) above.

Further, we claim:

(14) $([x][y])[z] = [x]([y][z])$

(15) if $x \sim e$ and $e \in E$, then $x \in E$

(16) $E$ forms one equivalence class, that is $E \in G$.

Property (14) is trivial. (15) follows from (7); (16) from (11) and (15). We write $\varepsilon = E$ when $E$ is seen as an *element* of $G$.

(17) $^{\forall}\xi \in G,\ \varepsilon\xi = \xi\varepsilon = \xi$

(18) $^{\forall}\xi \in G,\ \exists\eta \in G;\ \xi\eta = \eta\xi = \varepsilon$

Now, (17) follows by definition; (18) from (3) and (4). Thus, together with (14), $G$ is a group with elements $[x]$, where

(19) $[x] = \{ex;\ e \in E\}$ $(x \in S)$.

We have from (6) that

(20) if $e, f \in E$ and $e \neq f$, then $ex \neq fx$.

These properties (19) and (20) show that

(21) for each $x \in S$, the mapping $E \to [x]$, $e \mapsto ex$, is bijective.

Now we choose for every $\xi$ in $G$ an element $x_\xi$ of $S$ such that $\xi = [x_\xi]$ (using the axiom of choice), and construct a mapping $\varphi$: $E \times G \to S$ by putting $\varphi(e, \xi) = ex_\xi$. Note that $[\varphi(e, \xi)] = \xi$, so that by (21) $\varphi$ is also bijective.

Finally, to conclude the proof of the theorem, we claim:

(22) $\varphi$ is a semigroup isomorphism.

It suffices to show that $\varphi$ is a homomorphism. Let $(e, \xi)$ and $(f, \eta)$ be two elements of $E \times G$. Then we have

$$\varphi((e, \xi)(f, \eta)) = \varphi(ef, \xi\eta) = efx_{\xi\eta} = ex_{\xi\eta};$$

$$\varphi(e, \xi)\varphi(f, \eta) = ex_\xi fx_\eta = ex_\xi x_\eta.$$

But since $[x_\xi x_\eta] = [x_\xi][x_\eta] = \xi\eta = [x_{\xi\eta}]$, $x_\xi x_\eta = dx_{\xi\eta}$ for some $d \in E$. Thus $ex_\xi x_\eta = edx_{\xi\eta} = ex_{\xi\eta}$. □

Corollary. *The decomposition* $S \simeq E \times G$ *is unique up to isomorphism.*

**Reference**

Zassenhaus, The theory of groups, Chelsea, 1949.