

Flags and matchings

Yoshiaki UENO

To each (lower adjusted partial) flag of length k in an n -dimensional vector space over $GF(q)$ we assign an injective mapping ψ of $\{1, \dots, k\}$ into $\{1, \dots, n\}$ in such a way that exactly $q^{l(\psi)}$ of the flags map into a given mapping ψ , where $l(\psi)$ is the generalized "inversion number". The construction may also be regarded as an order-preserving mapping from the lattice of flags into the lattice of nests.

1. Introduction

Let $GF(q)$ denote the finite field with q elements. For a positive integer n , $V_n(q)$ denotes the n -dimensional vector space over $GF(q)$, $L_n(q)$ the lattice of subspaces of $V_n(q)$, $[1, n]$ the n -set $\{1, 2, \dots, n\}$ and \mathbf{B}_n the lattice of subsets of $[1, n]$. Here, both $L_n(q)$ and \mathbf{B}_n are ordered by inclusion, and rank in $L_n(q)$ and \mathbf{B}_n is dimension and cardinality, respectively.

It is well-known [see e.g. 3, p.21] that the number of complete flags in $V_n(q)$, $(1+q)(1+q+q^2)\dots(1+q+\dots+q^{n-1})$, can also be written as $\sum_{\sigma} q^{l(\sigma)}$, where σ runs through all permutations on the set $[1, n]$ and $l(\sigma)$ is the number of inversions in σ . In this paper we shall consider what would happen if we replace in this circumstance the set of permutations on a finite set by the set of injections between finite sets (The case of surjections is treated in [4]). This will be done in terms of introducing the lower adjusted partial flag and generalizing the concept of the inversion. A further refinement is also made by imposing some Young diagrammatic condition to the mappings.

We hope that this investigation will be the first step in the attempt of giving an explanation of the

well-known non-formal statement among combinatorists that the n -point set is "the n -dimensional vector space over the (non-existent) one-element field".

Acknowledgement. I wish to express my deep gratitude to Professor Nagayoshi Iwahori of the University of Tokyo for his valuable suggestions and encouragement throughout the preparation of this paper.

2. Definitions and results

Let k and n be integers such that $1 \leq k \leq n$.

Definition 2.1. A sequence (U_1, U_2, \dots, U_k) of elements of $L_n(q)$ is called a *lower adjusted (partial) flag* (abbrev. *LAF*) of length k if it satisfies:

- (1) $\dim U_i = k+1-i$ ($1 \leq i \leq k$);
- (2) $U_1 \supset U_2 \supset \dots \supset U_k$

We put $U_{k+1} = \{0\}$ for convenience' sake.

A sequence of integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ is a partition if $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k > 0$. We represent the elements of $V_n(q)$ as n -tuples (x_1, \dots, x_n) of elements of the field, and denote by W_d the d -dimensional subspace of $V_n(q)$ consisting of all the vectors $\mathbf{u} = (u_1, \dots, u_n)$ such that $u_j = 0$ for all $j > d$. Now, let λ be a partition such that $\lambda_1 = n$. The chain $W_{\lambda_1} \supset W_{\lambda_2} \supset \dots \supset W_{\lambda_k}$ in $L_n(q)$ we call the reference flag w.r.t. λ .

Definition 2.2. An LAF (U_1, U_2, \dots, U_k) is

subordinate to λ if $U_i \subset W_{\lambda_i}$ for $1 \leq i \leq k$.

We denote by $F_\lambda(q)$ the set of all LAF's subordinate to λ , and $f_\lambda(q)$ its cardinality.

It is easy to verify that $f_\lambda(q)$ is nonzero if and only if $\lambda_i \geq k+1-i$ ($1 \leq i \leq k$), and we have a complete explicit formula

$$f_\lambda(q) = [\lambda_k] [\lambda_{k-1}-1] \dots [\lambda_1-k+1],$$

where we put $[i] := (1-q^i)/(1-q)$ for i integer.

One consequence of this formula is that $f_\lambda(q)$ is a monic polynomial in q with non-negative integral coefficients. Such an observation suggests the possibility of giving a combinatorial interpretation of its coefficients, and thus proving this fact directly. Indeed this is done using the notion of the lower adjusted nest (LAN), whose definition goes parallel to that of the LAF as follows:

Definition 2.3. A sequence (A_1, A_2, \dots, A_k) of elements of \mathbf{B}_n is called a *lower adjusted nest* of length k if it satisfies:

- (1) $\# A_i = k+1-i$ ($1 \leq i \leq k$);
- (2) $A_1 \supset A_2 \supset \dots \supset A_k$

We put $A_{k+1} = \phi$ for convenience' sake.

Now, for given k, n and λ as above, the reference nest w.r.t. λ is by definition the chain $N_{\lambda_1} \supset N_{\lambda_2} \supset \dots \supset N_{\lambda_k}$ in \mathbf{B}_n (possibly with repetition) given by $N_d = \{1, 2, \dots, d\}$ ($1 \leq d \leq n$).

Definition 2.4. An LAN (A_1, A_2, \dots, A_k) is subordinate to λ if $A_i \subset N_{\lambda_i}$ for $1 \leq i \leq k$.

F_λ denotes the set of all LAN's subordinate to λ , an f_λ its cardinality. Then

Theorem 2.5. $f_\lambda = f_\lambda(1)$.

To prove 2.5. directly, we shall construct a mapping of $F_\lambda(q)$ into F_λ , and show that the cardinality of the inverse image of each LAN under this mapping is a power of q . A prototype of this kind of mapping was first developed by Knuth [1], (see also Milne [2]).

Now injections between finite sets come into picture as follows:

Definition 2.6. A *matching on λ* is an injection ψ of $[1, k]$ into $[1, n]$ such that $\psi(i) \leq \lambda_i$ ($1 \leq i \leq$

k).

A matching is sometimes called complete matching in the literature.

Proposition 2.7. *There exists a 1-1 correspondence between F_λ and the set of all matchings on λ .*

Composing this bijection and the mapping of $F_\lambda(q)$ onto F_λ , we have:

Theorem 2.8. $f_\lambda(q) = \sum q^{l(\psi)}$, where ψ runs through all matchings on λ .

Here, $l(\psi)$ is the natural generalization of the number of *inversions* in a permutation, defined as follows (*hook-empty condition*):

Definition 2.9. For an injection of $[1, k]$ into $[1, n]$, $l(\psi)$ is the number of nodes (i, j) on the Young diagram λ (i.e. $1 \leq i \leq k$ and $1 \leq j \leq \lambda_i$), such that $\psi(i) < j$ and $[i, k] \cap \psi^{-1}(j) = \phi$.

Thus the coefficient of q^d in $f_\lambda(q)$ is exactly the number of matchings on λ which have d "inversions".

3. System of Basis Vectors

Let $U = (U_1, \dots, U_k)$ be an LAF in $V_n(q)$ subordinate to λ . For each i we define $a_i \in [1, n]$, $A_i \in \mathbf{B}_n$, $\mathbf{v}_i \in V_n(q)$ and $B_i \subset V_n(q)$ by the following procedure:

(0) Put $A_{k+1} = B_{k+1} = \phi$, and $U_{k+1} = \{0\}$.

(1) Let $1 \leq i \leq k$. Then U_{i+1} is a codimension 1 subspace of U_i . We put

$$a_i := \max \{j \mid \exists (u_1, \dots, u_n) \in U_i, \\ u_l = 0 \text{ for all } l \in A_{i+1}, u_j \neq 0\};$$

$$A_i := A_{i+1} \cup \{a_i\} \text{ (disjoint union)};$$

$$\mathbf{v}_i = (u_1, \dots, u_n) \text{ is the unique vector in } U_i \\ \text{such that } u_l = 0 \text{ for all } l \in A_{i+1} \text{ and } u_{a_i} = 1;$$

$$B_i := B_{i+1} \cup \{\mathbf{v}_i\}.$$

Then B_i is a basis of U_i . We call $(\mathbf{v}_1, \dots, \mathbf{v}_k)$ the system of basis vectors (abbrev. SBV) of the LAF U . The matrix

$$\begin{matrix} v_{11} & v_{12} & \dots & v_{1n} \\ v_{21} & v_{22} & \dots & v_{2n} \\ \dots & \dots & \dots & \dots \\ v_{k1} & v_{k2} & \dots & v_{kn} \end{matrix}$$

where $\mathbf{v}_i = (v_{i1}, \dots, v_{in})$, is called the SBV-matrix of the LAF U . Note that its component $v_{ij} = 0$ unless $j \leq \lambda_i$; thus an SBV-matrix can be seen as a Young diagram whose entries are filled in with elements of $GF(q)$.

The mapping $: [1, k] \rightarrow [1, n], i \mapsto a_i$ is a matching on λ , while (A_1, \dots, A_k) is an LAN subordinate to λ , such that $A_i = \{a_i, a_{i+1}, \dots, a_k\}$. It is obvious that by this relation matchings on λ and LAN's subordinate to λ correspond bijectively.

Thus, to each LAF subordinate to λ we have assigned a matching on λ and an LAN subordinate to λ , which correspond with each other. Note also that the LAF is completely determined by its SBV-matrix, for $U_i = \langle \mathbf{v}_i, \mathbf{v}_{i+1}, \dots, \mathbf{v}_k \rangle_{GF(q)}, 1 \leq i \leq k$.

We now consider the number of ways one can construct a sequence of k vectors $(\mathbf{v}_1, \dots, \mathbf{v}_k)$ which is an SBV of some LAF which maps into a given matching ψ on λ . This is done by the following procedure:

Let $\Sigma = \{0, 1, *\}$ be a set of symbols. For a given matching ψ on λ we construct a Young tableau of shape λ , called the SBV-tableau, with its entries in Σ as follows:

(1) Fill in with 1 all boxes $(i, \psi(i))$, $1 \leq i \leq k$, of the Young diagram λ ;

(2) For each i , fill in with 0 all boxes (i, j) with $1 \leq j < \psi(i)$, and $(l, \psi(i))$, with $1 \leq l < i$;

(3) Fill in with $*$ all the boxes of λ not yet filled

in.

Now, an SBV-matrix is obtained if one replaces each symbol $*$ with an arbitrary element of $GF(q)$. Thus the number of ways one can construct an SVB for the given matching ψ is $q^{l(\psi)}$, where $l(\psi)$ is the number of $*$'s in the SBV-tableau. If $n = k$, so ψ is a bijection, then $l(\psi)$ is just the number of inversions in ψ regarded as a permutation on $[1, n]$. This concludes the proof of 2.8., and thus also 2.5.

Example. If $\lambda = (5 \ 5 \ 4 \ 2)$ and $\psi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 1 & 4 & 2 \end{pmatrix}$, then the SBV-tableau of ψ is

```

0 0 0 0 1
1 0 * 0 *
0 0 0 1
0 1

```

so that $l(\psi) = 2$.

REFERENCES

- 1) D. E. Knuth, Subspaces, Subsets, and Partitions, Journal of Combinatorial Theory 10 (1971) 178-180.
- 2) S. C. Milne, Mappings of Subspaces into Subsets, Journal of Combinatorial Theory (A) 33(1982) 36-47.
- 3) R. P. Stanley, Enumerative Combinatorics, Vol I, Birkhäuser, 1986.
- 4) Y. Ueno, Branching Flags, Branching Nests and Reverse Matchings, preprint, 1986.