An analogue of Mordell conjecture over function fields

Kazuhisa MAEHARA

Let f: $X \to C$ be a proper surjective morphism from a non-singular projective variety onto a non-singular curve defined over the complex number field. Let $P(\Omega_X^{\otimes d})$ be a projective bundle over X with $d = \dim X$. Assume that the fundamental sheaf $\mathcal{O}(1)$ on $P(\Omega_X^{\otimes d})$ is C-ample and that sections C_λ of f are Zariski dense in X. Then we prove that var f=0. This is a special case of higher dimensional Mordell conjecture over the function field.

Introduction

The Mordell conjecture is recently proven by Faltings. Faltings' Theorem implies that the number of rational points of an algebraic curve C defined over the rational number field Q with genus greater than one is finite. If C is defined over function field, the conjecture is true, as shown by J. Manin (1963), H. Grauert (1963), P. Samuel (1966) and M. Miwa (1966). S. Lang and E. Bombieri have conjectured that the Mordell conjecture be still true if V is a variety of general type. German, Noguchi, M. Deschamps and K. Maehara showed when V has the ample cotangent bundle over function field. Noguchi extended it when V is a hyperbolic manifold over the function field.

Result. Let f: $X \to C$ be a surjective morphism of projective varieties over the complex number field. Assume that a general fibre X_t is a variety of general type and that C is a curve. Suppose f: $X \to C$ has so many sections $\sigma_\lambda : C \to X$ that the $C_\lambda = \sigma_\lambda(C)$ are Zariski-dense in X. Let Ω_X denote the sheaf of the Kaehler differentials. Constructing the projective bundle p: $P(\Omega_X^{\otimes d}) \to X$, we assume that the fundamental sheaf $O(1)$ is $f \cdot p$-ample.

Theorem 1. *Suppose that $f \cdot p : P(\Omega_X^{\otimes d}) \to X \to C$ satisfies the above assumption. Then $f : X \to C$ is birationally equivalent to a trivial product over C.*

Boundedness of curves C_λ in the projective bundle $P(\Omega_X^{\otimes d})$. From the universality of the projective bundle and the natural surjection $\Omega_X^{\otimes d}|_{C_\lambda} \to \Omega_{C_\lambda}^{\otimes d}$, we have the unique section $s_\lambda : C_\lambda \to P(\Omega_X^{\otimes d})$.

Thus the intersection $(\mathcal{O}(1), s_\lambda(C_\lambda)) = (2g(C)-2)d$. From assumption $\mathcal{O}(1)$ is $f \cdot p$-ample, there exists an ample invertible sheaf H on C such that $\mathcal{O}(1) \otimes (f \cdot p)^* H$ is ample. The intersection $(\mathcal{O}(1) \otimes (f \cdot p)^* H, s_\lambda(C_\lambda))$ is independent of λ.

Put $L = \mathcal{O}(1) \otimes (f \cdot p)^* H$. The function $\varphi(t) = \chi(C_\lambda, L^{\otimes t}) = \deg_{C_\lambda} L^{\otimes t} + 1 - g(C) = t(L, C_\lambda) + 1 - g(C)$ which is independent of λ. Let H denote the Hilbert scheme of P with the Hilbert polynomial $\varphi(t)$. The universal family $U \subset P \times H$ contains all the sections im s_λ. Since U/H is flat, the intersection $(C_t, f^* \mathcal{O}(\text{one point})) = 1$ is independent of λ. Thus we can find a non empty open subset T of H such that $U|_T \to T$ is smooth and all fibres are isomorphic to C. Hence getting T smaller we have a non empty open set T_1 s.t. $U|_{T_1} \cong C \times T_1$. For simplicity we replace T by T_1. Then we obtain the following commutative diagram: $C \times T \cong U|_T \subset P \times T \to X \times T \to X \to C$, i. e.,

(Received Sep. 24, 1986)

Compactifying T into S, one has a dominant rational map. From the semipositivity of $f_*\omega_{X/C}^{\otimes m}$ and [M], the image of the f-pluricanonical mapping $X/C \to P(f_*\omega_{X/C}^{\otimes m})/C$ for $m \gg 0$ is a trivial product over C. Thus $f: X \to C$ is birationally equivalent to a trivial product over C.

Q.E.D.

Reference

[M], Maehara, K.: Finiteness property of varieties of general type. Math. Ann. 262, 101-123 (1983)