Non Commutative Geometry 1

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Abstract

In this article we shall introduce a non commutative algebraic geometry by Kontsevich and Rosenberg ([Kon], [Mch]) and represent it by recently developed theory of corings and comodules ([Brz]). We restrict ourselves to the category of non commutative algebraic varieties and develop the birational geometry by infinite Galois theory of skew fields making use of profinite groups ([AM], [BJ], [Breen1], [Breen2], [Gir], [SGA], [S1], [S2], [Shatz], [Se], [Zuo], [RBZL]). We apply it to non commutative varieties of general type defined later over the field of characteristic 0 ([Iita], [Fuj], [Kaw], [Mats], [MP], [Km3]). Main tools of classification of projective varieties ([Iita], [Mum], [Vieh], [Ko1], [Zuo]) are so called characteristic $p > 0$ technic ([MP], [Ko2]), [BBD], [Berth]) and weak positivity direct images of multi-power of dualizing sheaves for fibre spaces ([Kaw], [Ws], [Vieh], [Nak], [Km1]) as well as Kawamata-Viehweg vanishing theorems ([MP]). Instead of these tools, we make use of profinite groups.

1 Introduction:

In this section we consider the corings ([Brz]) that have a grouplike element $g$ which are related to ring extensions $B \to A$. Throughout this section $C$ denotes an $A$-coring. Galois corings are isomorphic to the Sweedler coring associated to a ring extension $B \to A$ induced by the existence of a grouplike element. The following theorem determines when the $g$-coinvariants functor is an equivalence.

Theorem 2. Let $g$ be a grouplike element of $C$, $B = A^g_{coC}$, and $G_g : M^C \to M_B \quad M \mapsto M_g^{coC}$ the $g$-coinvariants functor.

1. The following statements are equivalent:

(i) $(C, g)$ is a Galois coring and $A$ is a flat left $B$-module.
(ii) $AC$ is flat and $A_g$ is a generator in $MC$.

2. The following statements are equivalent, too.

(i) $(C, g)$ is a Galois coring and $_BA$ is faithfully flat.

(ii) $AC$ is flat and $A_g$ is a projective generator in $MC$.

(iii) $AC$ is flat and $\text{Hom}^C(A_g, -) : MC \to MB$ is an equivalence whose inverse is $- \otimes_B A : MB \to MC$.

The theorem above is a restatement of one of the main results in non commutative descent theory([HS], [S1], [S2], [Km2]). In fact, for an algebra extension $B \to A$, there exists a comparison functor $- \otimes_B A : MB \to \text{Desc}(A/B)$ which to each right $B$-module $M$ gives a descent datum $(M \otimes_B A, f)$ with $f : M \otimes_B A \to M \otimes_B A \otimes_B A, m \otimes a \mapsto m \otimes 1_A \otimes a$.

If $(C, g)$ is a Galois coring, then the category of right $C$-comodules is isomorphic to the category of descent data $\text{Desc}(A/B)$. Thus if $B \to A$ is faithfully flat, then it is an effective descent morphism. Furthermore, Galois corings correspond to comparison functors that are equivalences. Note that if $B \to A$ is a faithful flat extension, then $(A \otimes_B A, 1_A \otimes_B 1_A)$ is a Galois coring. The objects in the category of corings are pairs $(C : A)$, where $A$ is an $R$-algebra and $C$ is an $A$-coring. A morphism between corings $(C : A)$ and $(D : B)$ is a pair of mappings $(\gamma : \alpha) : (C : A) \to (D : B)$ satisfying

1. $\alpha : A \to B$ is an algebra map. Hence $D$ is considered to be an $(A, A)$-bimodule.

2. $\gamma : C \to D$ is a map of $(A, A)$-bimodules such that

$$\xi \circ (\gamma \otimes_A \gamma) \circ \Delta_C = \Delta_C \circ \gamma ; \xi_D \circ \gamma = \alpha \circ \xi_C,$$

where $\xi : D \otimes_A D \to D \otimes_B D$ is the canonical map of $(A, A)$-bimodules.

Since an algebra $A$ can be considered as a trivial $A$-coring $(A : A)$, this category of corings contains the category of $R$-algebras.

Left $C$-comodule is defined as a left $A$-module $M$, with a coassociative and counital left $C$-coaction. $C$-morphisms between left $C$-comodules $M, N$ are defined in an obvious way. Left $C$-comodules and their morphisms form a pre-additive category $^C M$.

3 Geometric View

Let $k$ be a commutative field and $A, B$ $k$-algebras. The objects of the opposite category of corings denote Spec $(C : A)$ and a morphism between Spec $(D : B) \to$ Spec $(C : A)$ denotes Spec $(\gamma : \alpha)$. This category is said to be that of covers. Furthermore, the
category $\mathcal{C} M$ is abelian and it is denoted $QCoh(\text{Spec}(C : A))$. The canonical morphism $f : \text{Spec}(B \otimes_A B : B) \to \text{Spec}(A : A)$ defines an equivalence between abelian categories $f^* : QCoh(\text{Spec}(A : A)) \cong QCoh(\text{Spec}(B \otimes_A B : B))$. Owing to Morita-Takeuchi theorems or Grothendieck ideas, the geometry of covers consist in $QCoh(\text{Spec}(C : A))$.

The cover $\text{Spec}(C : A)$ equipped with an epimorphism $A \otimes A \to C$ which is a morphism of coalgebras is said to be a space cover. A morphism in the category of space covers is defined to be a morphism as covers compatible with additional structure as space covers. Let $f = (\gamma, \alpha), g = (\delta, \beta)$ be two morphisms between space covers $\text{Spec}(C : A) \to \text{Spec}(D : B)$. When for $x_i \otimes y_i \in \ker(A \otimes A \to C)$, the following equation holds $\sum_i \alpha(x_i) \cdot \beta(y_i) = \beta(x_i) \cdot \alpha(y_i) = 0$ in $B$, two morphisms $f$ and $g$ are defined to be equivalent.

**Definition 4.** The category of non commutative algebraic spaces over $k$ is the localization category with the canonical morphisms invertible of the quotient of the category of space covers by equivalence of equivalent morphisms.

The category of separated quasi-compact schemes over $k([\text{Gir}], [\text{SGA}], [\text{GG}], [\text{HS}], [\text{Kato}], [\text{KKMS}])$ and the opposite category of that of $k$-algebras are equivalent to a full subcategory of the category of non commutative algebraic spaces over $k([\text{Kon}], [\text{Cohn}])$, respectively. The category of non commutative algebraic spaces over $k$ admits finite limits. A non commutative algebraic space of the type $\text{Spec}(A : A)$, where $A$ is a $k$-algebra, is said to be an affine space. Let $\mathbb{NP}^d_k$ be the non commutative projective space over $k$ and $A$ a $k$-algebra. The set $\text{Hom}(\text{Spec}(A : A), \mathbb{NP}^d_k)$ is the set of quotient modules of $A^d$ which are locally free $A$-modules of dimension 1 in flat topology([SGA]). In the same way, we have the non commutative Grassmannian $NGr_k(r, d)([\text{Kon}], [\text{Laum}])$.

## 5 Extension of skew fields and Galois theory

Let $A$ be an integral domain such that $xA \cap yA \neq 0$ for $x, y \in A$, which is called a right Ore domain. Let $S = R^\times$. Then the localization of $A$ at $S$ is a skew field $K = A_S$ and the natural homomorphism $\lambda : A \to K$ is a monomorphism. Recall that every ring with a homomorphism to a field has invariant basis number. From now on, we treat a non commutative algebraic space of the type $\text{Spec}(C : A)$ where $A$ is a Ore domain. Any equation of degree $n > 0$, $x^n + a_1x^{n-1} + \cdots + a_n = 0$ ($a_i \in K$), has a right root in some extension of $K$. There exists the right algebraic closure $\overline{K}$ over $K$ such that any equation of the type above has a right root in $\overline{K}$. A Galois extension $L/K$ is outer if and only if the centralizer of $K$ in $L$ is just the centralizer of $L$. Let $k$ be a commutative field of characteristic 0 and $K$ a $k$-algebra of finite type, skew field. Let $\overline{K}$ be the right algebraic
closure of $K$ such that the centralizer of $K$ in $\overline{K}$ is just the centralizer of $\overline{K}$ ([Cohn]). Let $(K_i)_{i \in I}$ be a family of skew fields such that

1. $K_i$ are subfields of $K$,
2. $K_i$ are $k$-algebras of finite type,
3. the centralizers of $K_i$ in $\overline{K}$ are the center of $\overline{K}$.

Then the $\overline{K}/K_i$ are all outer Galois extensions, whose Galois groups are profinite groups. We need Jacobson-Bourbaki correspondence ([Cohn], [BJ]): Let $K$ be a field and $End(K)$ the endomorphism ring of the additive group $K^+$ with the finite topology. We have an order-reversing bijection between the subfields $D$ of $K$ and the closed $K$-subrings of the type $End_{D-}(K)$ of $End(K)$. From this, we have the following Galois connection: Let $L/K$ be an algebraic Galois extension with Galois group $G$ outer. Then we have a bijection between intermediate fields $D$, i.e., $K \subset D \subset L$ and the closed subgroups $H$.

6 Non commutative algebraic birational geometry

We investigate the non commutative algebraic birational geometry from the point of view of the profinite Galois groups ([Gir], [Breen1], [Breen2]). Let $X \to S$ be a non commutative fibre space of algebraic spaces over Spec ($k$), with the generic point of the generic general fibre one of skew fields $K_i$ which are defined in the preceding section ([Mch], [RBZL]). Let $1 \to G \to E \to P \to 1$ be an extension of a profinite group $P$ by a profinite group $G$ associated to the non commutative fibre space $X \to S$. Hence $G$ is a profinite group, that is one of the Galois group $Gal(\overline{K}/K_i)$. To an exact sequence $1 \to \text{Inn}G \to \text{Aut}G \to \text{Out}G \to 1$, we have an exact sequence

$$H^1(P, \text{Inn}G) \to H^1(P, \text{Aut}G) \to H^1(P, \text{Out}G),$$

i.e.,

$$\text{Hom}(P, \text{Inn}G) \to \text{Hom}(P, \text{Aut}G) \to \text{Hom}(P, \text{Out}G).$$

Here Out$G$ denotes the outer automorphism group of $G$. A group extension is an element of $H^1(P, G \to \text{Aut}G)$ , where $G \to \text{Aut}G$ is a crossed module. We have

$$1 \to H^2(P, Z(G)) \to H^1(P, G \to \text{Aut}G) \to H^1(P, \text{Out}G).$$

Here $Z(G)$ denotes the center of $G$. Assume that Out($G$) is an algebraic group of countable connected components. Then the canonical representation $\rho : P \to \text{Out}G$ turns out to be trivial after replacing a profinite group associated to a finite morphism $S' \to S$ in the
following lemma. Furthermore assume that the extension is neutral. This assumption is satisfied since there exists a homomorphism from $1 \to G' \to G' \times P \to P \to 1$ to $1 \to G \to E \to P \to 1$, where $P \to P$ is an identity, $G' = \text{Gal}(\overline{K}/K)$.

Since we have $H^2(P, Z(G)) \to H^1(P, G \to \text{Aut}(G))$, the extension $1 \to G \to E \to P \to 1$ is given by pushing out an extension $1 \to Z(G) \to E' \to P \to 1$. Hence $E'$ is a semi-direct product $Z(G) \rtimes P$, which is contained in a semi-direct product $G \rtimes P$. Thus this central extension is trivial. Therefore by pushing out this central extension, the extension $1 \to G \to E \to P \to 1$ is trivial.

**Lemma 7.** There exists a homomorphism $P' \to P$ with $(P' : P) < \infty$ such that the representation $\rho : P' \to \text{Out}(G)$ is trivial. Here $P'$ denotes the absolute Galois group $\text{Gal}(\overline{R(S')}/R(S'))$.

**Proof.** Let $A$ denote $\text{Out}(G)$. This group $A$ is locally algebraic([SGA]). The natural representation $\rho : P \to A$ induces $\overline{\rho} : P \to A/A^0$, where $A^0$ denotes the neutral component of $A$. There is no countable profinite group. Since $A/A^0$ is a countable set, $\overline{\rho}(P)$ is a finite group. Replace by $P$ the kernel of $\overline{\rho}$. We have $\rho : P \to A^0$. Hence we have an isomorphism

$$H^1(R(S)/R(S), A^0(R(S)) \cong H^1(BP, A^0).$$

Let $P$ be an $A^0$-torsor associated to $\rho : P \to A^0$. $A^0$ is algebraic (quasi-compact, faithfully flat and of finite type) over $\text{Spec}(R(S))$. Thus there exists a generically finite $S' \to S$ such that an $A^0$-torsor $P$ is trivial over $\text{Spec}(R(S'))$. Hence the representation $\rho : P' \to \text{Out}(G)$ is trivial. \hfill \Box

Thus we obtain the following result in our proof.

**Theorem 8.** Let $1 \to G \to E \to P \to 1$ be an extension of a profinite group $P$ by a profinite group $G$. Assume

(a) $\text{Out}(G)$, is an algebraic group with countable connected components.

(b) $E \to P$ has a section which is a group homomorphism, i.e., a neutral extension.

Then there exists a profinite group $P'$ such that the pull-back of the extension $1 \to G \times_P P' \to E \times_P P' \to P' \to 1$ is a direct product.

Let $X$ be a non commutative fibre space of smooth varieties over $\text{Spec} \ k$. We have the canonical homomorphism $\Gamma(X, \Omega_{X}^{\otimes m}) \otimes \mathcal{O}_X \to \Omega_{X}^{\otimes m}$. Assume this homomorphism is generically epimorphism. Then it determines a map from an open of $X$ to non commutative Grassmannian[Kon]). When this map is birational, i.e., the field defined by the generic point of $X$ and that of the image are isomorphic, the assumption (a) above is satisfied.
Remark 9. Let $\phi : G_1 \to G_2$ be an open continuous homomorphism of profinite groups. $\phi(G_1) \subset G_2$. Let $Z(G_2)C_{\phi(G_1)}(\phi(G_1))$ denote $C$. Then for a homomorphism between extensions of $P$ by $G_1$ and $G_2$ respectively, one has homomorphisms $H^2(P, Z(G_1)) \to H^2(P, Z(G_2))$. There exists an open subgroup $P'$ of finite index of $P$ such that $H^2(P', Z(G_2)) \to H^2(P', C)$ is injective.

References


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