Dynamics of maps on $\mathbb{R}^2$ associated with unicritical polynomials

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The dynamics of maps on $\mathbb{R}^2$ associated with unicritical polynomials is considered. Structure of the interior of the filled Julia set is investigated.

1 Quadratic maps

In this section, we consider a quadratic map on $\mathbb{R}^2$ of the form :

$$f_c(x,y) = (x^2 + y^2 + c, 2xy), \quad c < 1/4.$$

It appears in the dynamics of quadratic maps on coquaternions. If we put $\phi(x,y) = (x+y, x-y)$ and $g_c(x,y) = (p_c(x), p_c(y))$, where $p_c(x) = x^2 + c$ is a unicritical quadratic polynomial, then it follows

$$\phi \circ f_c = g_c \circ \phi.$$

Thus the affine map ϕ conjugates the map f_c to a simpler map g_c and the dynamics of f_c is translated from that of g_c. For example, since $K(g_c) = K(p_c) \times K(p_c) = [-\beta, \beta] \times [-\beta, \beta]$, we have

$$K(f_c) = \{(x,y) \in \mathbb{R}^2; |x+y|, |x-y| \le \beta\}.$$

Let $\alpha = \dfrac{1-\sqrt{1-4c}}{2}$ and $\beta = \dfrac{1+\sqrt{1-4c}}{2}$ be the fixed points of p_c. Then β is always repelling. If $-3/4 < c < 1/4$, α is the attracting fixed point of p_c

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and $intK(p_c) = (-\beta, \beta)$ is the basin of α. That is, it is the set of points whose orbits converge to α.

In case $-5/4 < c < -3/4$, p_c has an attracting 2-cycle $\gamma_\pm = \dfrac{-1 \pm \sqrt{-4c-3}}{2}$, whose basin $\mathcal{B}$ is $(-\beta, \beta) \setminus \cup_{n\geq 0} p_c^{-n}(\alpha)$. This basin consists of infinitely many connected components. Let $U_\pm$ be the connected component containing $\gamma_\pm$ respectively. Then $\mathcal{B}$ is the union of $\mathcal{B}_\pm = \cup_{n\geq 0} p_c^{-2n}(U_\pm)$. Then g_c has two attracting 2-cycles $\{(\gamma_\pm, \gamma_\pm)\}$ and $\{(\gamma_\pm, \gamma_\mp)\}$. The corresponding attracting 2-cycles of f_c are respectively $\{(\gamma_\pm, 0)\}$ and $\{(-1/2, \pm\sqrt{-4c-3}/2)\}$. Thus, $g^{2n}(x, y) \to (\gamma_*, \gamma_{**})$ if $(x, y) \in \mathcal{B}_* \times \mathcal{B}_{**}$, where $*$ and $**$ indicate $+$ or $-$.

From these arguments,we have the following.

Theorem 1. *Suppose* $-5/4 < c < -3/4$.
If $(x + y, x - y) \in \mathcal{B}_\pm \times \mathcal{B}_\pm$, *then* $f_c^{2n}(x, y) \to (\gamma_\pm, 0)$.
If $(x + y, x - y) \in \mathcal{B}_\pm \times \mathcal{B}_\mp$, *then* $f_c^{2n}(x, y) \to (-1/2, \pm\sqrt{-4c-3}/2)$.

Figure 1 indicates the basin of f_c with $c = -1$. The white regions correspond to $\mathcal{B}_\pm \times \mathcal{B}_\pm$, while the black regions correspond to $\mathcal{B}_\pm \times \mathcal{B}_\mp$.

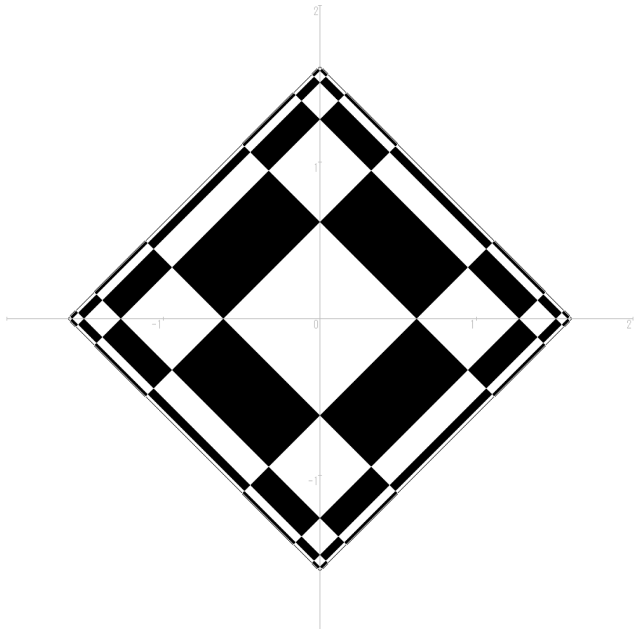

Figure 1: Basin of f_c with $c = -1$

Here we use the basic facts in complex dynamcs, which is seen in Milnor [M].

2 Maps of higher degrees

The same argument works for higher degree polynomial maps of the form :

$$f_c(x,y) = \left(\frac{(x+y)^d + (x-y)^d}{2} + c, \frac{(x+y)^d - (x-y)^d}{2} \right), \quad d \geq 2.$$

Put $p_c(x) = x^d + c$, $g_c(x,y) = (p_c(x), p_c(y))$ and $\phi(x,y) = (x+y, x-y)$. Then

$$\phi \circ f_c = g_c \circ \phi.$$

The dynamics of p_c highly depends on whether the degree d is even or odd. This is because p_c is monotonely increasing if and only if d is odd. Put $c_0 = \dfrac{d-1}{d^{d/(d-1)}}$.

Suppose that d is odd. Then the following holds:

(1) p_c has a unique real fixed point for $c > c_0$.
(2) p_c has a double fixed point β at $c = c_0$.
(3) p_c has three real fixed points for $c \in (-c_0, c_0)$.
(4) p_c has a double fixed point $-\beta$ at $c = -c_0$.
(5) p_c has a unique real fixed point for $c < -c_0$.

Suppose that d is even. Then the following holds:

(6) p_c has no real fixed points for $c > c_0$.
(7) p_c has a double fixed point β at $c = c_0$.
(8) p_c has two real fixed points for $c < c_0$.

In the sequel, we assume that $-c_0 < c < c_0$ when d is odd or $c < c_0$ when d is even. Let $\beta > 0$ be the maximal real fixed point of p_c, which is known to be repelling.

Lemma 1. *If d is odd, then $K(p_c) = [\beta', \beta]$, where $\beta' < 0$ is the minimal fixed point of p_c, which is known to be repelling.*

If d is even, then $K(p_c) \subset [-\beta, \beta]$. Moreover, suppose that $0 \in K(p_c)$. Then $K(p_c) = [-\beta, \beta]$.

Note that $0 \in K(p_c)$ holds if and only if $-c_0 \leq c \leq c_0$ in case d is odd. Suppose that d is even. Then $0 \in K(p_c)$ holds if and only if $-\beta \leq c$. Hence, there exists c_0' such that $0 \in K(p_c)$ if and only if $c_0' \leq c \leq c_0$.

Proof. First assume that d is even. Then $p_c^n(x) \to +\infty$ if $x > \beta$. If $x < -\beta$, $p_c(x) = p_c(-x) > \beta$, hence $p_c^n(x) \to +\infty$. Therefore, $K(p_c) \subset [-\beta, \beta]$. Note that $c = p_c(0)$ is the minimum value of p_c. Therefore, $p_c^n(0) \to +\infty$ if $c < -\beta$. If $c \geq -\beta$, then $p_c([-\beta, \beta]) \subset [-\beta, \beta]$ and we have $K(p_c) = [-\beta, \beta]$.

Next suppose d is odd. Then it is easy to see that $p_c^n(x) \to +\infty$ if $x > \beta$, while $p_c^n(x) \to -\infty$ if $x < \beta'$. Thus $K(p_c) \subset [\beta', \beta]$. Since p_c is monotonely increasing, $p_c([\beta', \beta]) \subset [\beta', \beta]$, which implies $K(p_c) = [\beta', \beta]$. □

Note that, if d is odd and $-c_0 < c < c_0$, p_c has an attracting fixed point α, whose basin is equal to $intK(p_c)$. This easily follows also from the fact that p_c is monotonely increasing.

The corresponding attracting fixed point of f_c is the point $\gamma = (\alpha, 0)$.

Theorem 2. *Suppose that d is odd and $-c_0 < c < c_0$. Then*

$$K(f_c) = \{(x, y) \in \mathbb{R}^2; \beta' \leq x + y, x - y \leq \beta\}.$$

Its interior $intK(f_c)$ is the basin of the fixed point γ of f_c.

Proof. Since $K(g_c) = [\beta', \beta] \times [\beta', \beta]$, the first assertion follows. The second one follows from the fact that $g_c^n(x, y) \to (\alpha, \alpha)$ for any $(x, y) \in intK(g_c)$. □

If d is even, as c decreases, period-doubling bifurcations appear. That is, there exists a decreasing sequence $c_0 > c_1 > c_2 > \cdots$ converging to c_∞ such that, for $c \in (c_{n+1}, c_n)$, p_c has an attracting cycle of period 2^n. Note that $c_0 = 1/4, c_1 = -3/4, c_2 = -5/4$ if $d = 2$.

Theorem 1 extends to any even degree. If $c \in (c_1, c_0)$, p_c has an attracting fixed point α and $K(g_c) = K(p_c) \times K(p_c) = [-\beta, \beta] \times [-\beta, \beta]$. Thus we have

$$K(f_c) = \{(x, y) \in \mathbb{R}^2; |x - y|, |x - y| \leq \beta\}.$$

If $c \in (c_2, c_1)$, p_c has an attracting 2-cycle $\{\gamma_1, \gamma_2\}$. Let α be another repelling real fixed point of p_c and let U_j be the connected component of $intK(p_c) \setminus \cup_{n \geq 0} p_c^{-n}(\alpha)$ containing γ_j respectively for $j = 1, 2$. Put $\mathcal{B}_j =$

$\cup_{n\geq 0} p_c^{-2n}(U_j)$. Then g_c has two attracting 2-cycles $\{(\gamma_1,\gamma_1),(\gamma_2,\gamma_2)\}$ and $\{(\gamma_1,\gamma_2),(\gamma_2,\gamma_1)\}$. The corresponding attracting 2-cycles of f_c are $\{(\gamma_1,0),(\gamma_2,0)\}$ and $\left\{\left(\frac{\gamma_1+\gamma_2}{2},\frac{\gamma_1-\gamma_2}{2}\right),\left(\frac{\gamma_1+\gamma_2}{2},\frac{\gamma_2-\gamma_1}{2}\right)\right\}$ respectively.

Then $g_c^{2n}(x,y)\to(\gamma_i,\gamma_j)$ if $(x,y)\in\mathcal{B}_i\times\mathcal{B}_j$ for $i,j=1,2$. Thus we have the following.

Theorem 3. *Suppose that d is even and $c_2<c<c_1$. Then the folowing hold for $i,j=1,2$.*
If $(x+y,x-y)\in\mathcal{B}_j\times\mathcal{B}_j$, then $f_c^{2n}(x,y)\to(\gamma_j,0)$.
If $(x+y,x-y)\in\mathcal{B}_j\times\mathcal{B}_{3-j}$, then $f_c^{2n}(x,y)\to\left(\frac{\gamma_1+\gamma_2}{2},\frac{\gamma_j-\gamma_{3-j}}{2}\right)$.

References

[M] J. Milnor, *Dynamics in one complex variable*. 3rd edition (Annals of Math. Studies, 160). Princeton University Press, Princeton, NJ, 2006.